Tomographic Image Reconstruction Using Few Projections

A Review on CS and Numerical Algorithms

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Overview of Reconstruction Problem

Beer-Lambert's law :

$$m(X) = N_0 \exp(-(AX)) \tag{1}$$

Physical model(Sampling under Poisson law) :

$$Y \sim \mathcal{P}(m(X)) \tag{2}$$

$$B = \log N_0 - \log Y \tag{3}$$

MAP Estimation :

$$\min_x - \log \mathbb{P}(y|x) - \log \mathbb{P}(x) \tag{4}$$

Approximation & Choice of prior

High dose + Edge-preservation reconstruction :

- Approximation of $-\log \mathbb{P}(y|x)$ by quadratic term
- Choice of regularization prior $TV(x)$

$$\min_x \frac{1}{2}\|Ax - b\|_2^2 + \lambda TV(x) \tag{5}$$

Constrainted optimization formulation :

$$\min_x TV(x) \text{ s.t. } \|Ax - b\|_2^2 \leq \varepsilon^2 \tag{6}$$

Good reconstruction quality observed with small number of projections for sparse object.

Sparsity Prior

TV semi-norm prompts "cartoon-like" object :

$$TV(x) := \sum_k |\nabla x_k| \quad (7)$$

$$TV(x) := \sum_k \sqrt{|\partial_x x_k|^2 + |\partial_y x_k|^2} \quad (8)$$

Sparsifying transform : basis, frames(redundant basis), or sub-basis.

- Natural basis : Spikes
- TV : Sharp edges/Time-sparsity
- DCT : Periodic smooth
- Wavelet : Texture...

Sparsifying transform : Analysis vs. Synthesis

- Analysis. For invertible transform(basis) :

$$x = D\alpha \Leftrightarrow D^{-1}x = \alpha \tag{9}$$

- Synthesis. For non invertible transform(frame, sub-basis) :

$$x = D\alpha \tag{10}$$

Transformed coefficients α are :

- S-sparse : only S non zero terms
- approximatively sparse : rapide decreasing rate $\longrightarrow$ sparse approximation error

Motivation

Medical objects are often sparse :

- Blood vessels : natural basis
- Tissues : wavelet
- Organs/Bones : TV
- ...

Medical measurements are expensive to take :

- X-ray : dose
- MRI : time

Taking advantage of sparsity to reduce measurements ?

Find the sparse solution(s)

Measurement matrix A (Radon transform) is underdetermined.

- MSE solution (ART, FBP)

$$\min_x \|x\|_2^2 \text{ s.t. } Ax = b \Leftrightarrow x^* = A^\dagger b \tag{11}$$

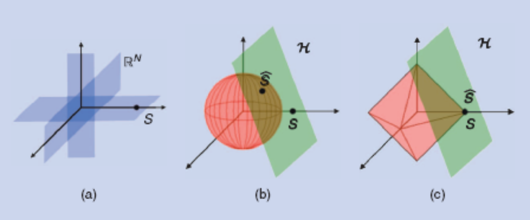
- Sparest solution

$$\min_x N(x) \text{ s.t. } Ax = b \tag{12}$$

Sparsity measure $N(x)$:

- Natural : $\|x\|_0 := \#\{k : x_k \neq 0\}$
- Nonconvex l_p norm : $\|x\|_p^p := \sum_k |x_k|^p$, for $0 < p < 1$
- Convex l_1 norm : $\|x\|_1 := \sum_k |x_k|$

Why use $l_p (0 \leq p \leq 1)$ norm as sparsity measure ?



Theorem (Decreasing rate of R-sphere vector)

Given $x \in \mathbb{R}^N$, *with* $\|x\|_p = R$, *and* $|x_1| \geq |x_2| ... \geq |x_N|$, *then*

$$|x_k| \leq R k^{-1/p}$$

- High dimension ball is **almost empty**.
- $l_p (p \leq 1)$ ball is **near the axis**.

Sparse reconstruction with l_p norm

The true object x is (app.) sparse under sparsifying transform :

$$\min_{\alpha} \|\alpha\|_p^p \text{ s.t. } D\alpha = x \tag{13}$$

Applied to sensing matrix A :

$$\min_{\alpha} \|\alpha\|_p^p \text{ s.t. } AD\alpha = b \tag{14}$$

Noisy case :

$$\min_{\alpha} \|\alpha\|_p^p \text{ s.t. } \|AD\alpha - b\|_2^2 \leq \varepsilon^2 \tag{15}$$

A, D are generally underdetermined.
Reconstruction : $x^* = D\alpha^*$

Reconstruction error & choice of l_p norm

Reconstruction error $\|x - x^*\|_2$ depends on :

- Uniqueness of sparse representation : Sparsifying transform D
- Dimension of solution space : Measurement matrix A
- Observation noise $\varepsilon = b - Ax$

Choice of l_p-norm$(0 \leq p \leq 1)$:

- Sparsity or approximatively sparsity
- Solution uniqueness condition : local optimum for $p < 1$
- Efficient numerical algorithm

Idea of CS

To recovery a S-sparse signal x with underdetermined sensing matrix A :

- P_0 problem(NP-hard) : $\min_x \|x\|_0$ s.t. $Ax = b$
- P_1 problem(convex) : $\min_x \|x\|_1$ s.t. $Ax = b$

Equivalence of P_0 and P_1 ?

Definition (Restricted Isometry Property)

RIP of A is the smallest $\delta_S > 0$ s.t. for all S-sparse signal x :

$$(1 - \delta_S)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \delta_S)\|x\|_2^2 \quad (16)$$

i.e., all S-submatrix of A is a "restricted" isometry.

P_0 and P_1 equivalence

Theorem (Perfect reconstruction)

Given that $2S$-RIP of A is $\delta_{2S} \le \sqrt{2} - 1$, for a S-sparse true signal x, P_1 solution x^* is **exactly** *x.*

Theorem (Almost Perfect reconstruction)

Given that $2S$-RIP of Φ is $\delta_{2S} \le \sqrt{2} - 1$, for a general true signal x, P_1 solution x^* obeys :

$$\begin{aligned} \|x^* - x\|_1 &\le C\|x - x_S\|_1 \quad \textit{and} \\ \|x^* - x\|_2 &\le C\frac{\|x - x_S\|_1}{\sqrt{S}} \end{aligned}$$

where $x_S = (x_1, x_2, ..x_S, 0..)$ is the S biggest term approximation of x.

Robustess

For noisy observation $b = Ax + \varepsilon$:

$$\min_x \|x\|_1 \ \text{ s.t. } \ \|Ax - b\|_2 \leq \varepsilon \qquad (NP_1)$$

Theorem (Robust reconstruction)

Under the same hypothesis, the solution x^* to (NP_1) obeys :

$$\|x^* - x\|_2 \leq C_1 \varepsilon + C_2 \frac{\|x - x_S\|_1}{\sqrt{S}} \qquad (17)$$

Random matrices

Sensing matrices A satisfy RIP with probability $1 - O(e^{-N})$:

- Gaussian matrix : $S \leq C.K/\log(N/K)$
- Binary matrix : $\mathbb{P}(A_{i,j} = \pm 1/\sqrt{K}) = 0.5$
- Fourier matrix : $S \leq C.K/\log N$ (conjectured)

K **randomly choosed observations can measure the essential part of** x **!**

P_0 and P_p equivalence

P_p problem :

$$\min_x \|x\|_p^p \ \text{ s.t. } \ Ax = b \tag{18}$$

Theorem (P_p robust reconstruction)

Given that following holds for some $k > 1$, and $kS \in \mathbb{Z}_+$:

$$\delta_{kS} + k^{2/p-1}\delta_{(k+1)S} < k^{2/p-1} - 1 \tag{19}$$

Then for arbitrary true signal x, the solution x^* to P_p obeys :

$$\|x^* - x\|_2^p \leq C_1\varepsilon^p + C_2\frac{\|x - x_S\|_p^p}{S^{1-p/2}} \tag{20}$$

Particularly, one gets P_1 results for $k = 3$ and $p = 1$.

Sparse reconstruction with TV

For partial Fourier measurement A, TV regularization equals to :

$$\min_{\alpha} \|\alpha\|_1 \text{ s.t. } A\alpha = b', \text{ with } \alpha = \nabla x \tag{21}$$

FIG.: P_1 reconstruction with 18 projections 15 projections and 13 projections. For the left one, P_p reconstruction with $p = 0.5$ achieves the same result using only 10 projections.

Problems & variant models

For general model :

$$\min_{\alpha} \|\alpha\|_p^p \ \text{ s.t. } \ AD\alpha = b \tag{22}$$

RIP property of sensing matrix A could be violated since :

- D is deterministic sparsifying transform
- Unique sparse representation doesn't imply unique sparse reconstruction

Analysis model(example) :

$$\min_{x} TV(x) + \lambda\|\mathcal{F}x\|_1 \ \text{ s.t. } \ Ax = b \tag{23}$$

No theoretical estimation available for the reconstruction error ?

Sparsifying transform

A **dictionary** $D = [d_1 ... d_M]$ of $\mathbb{C}^N$ satisfies :

$$\|d_m\|_2 = 1, \quad \text{and} \quad \mathrm{Span}(D) = \mathbb{C}^N \tag{24}$$

Find the sparest representation(s) when dictionary is **redundant**($M > N$) :

- $P_0 : \min_\alpha \|\alpha\|_0$ s.t. $D\alpha = x$
- $P_1 : \min_\alpha \|\alpha\|_1$ s.t. $D\alpha = x$

Equivalence : the **same unique** solution α^*.

P_0 and P_1 equivalence

Mutual Coherence of a dictionary :

$$\mu(D) \triangleq \max_{i \neq j} |\langle d_i, d_j \rangle| \tag{25}$$

P_0 P_1 equivalence conditions :

- General dictionary : $\|x\|_0 < \left(1 + \mu(D)^{-1}\right)/2$
- Union of 2 orthonormal bases : $\|x\|_0 < (\sqrt{2} - 0.5)\mu(D)^{-1}$
- Union of L orthonormal bases :

$$\|x\|_0 < \left(\sqrt{2} - 1 + \frac{1}{2(L-1)}\right)\mu(D)^{-1}$$

Incoherence between measurement and representation

RIP property remains true for general model :

$$\min_{\alpha} \|\alpha\|_1 \quad \text{s.t.} \quad AD\alpha = b \tag{26}$$

with overwhelming probability if :

$$K \geq \mu(A, D)^2 CS(\log N)^4 \tag{27}$$

where $\mu(A, D) := \max_{i,j} |\langle a_i, d_j \rangle|$.

Design the dictionary D to be incoherent with A.

Robust P_1 problem

Robust P_1 problem :

$$\min \ \|x\|_1 \ \text{ s.t. } \ \|Ax-b\|_2^2 \leq \varepsilon^2 \qquad (NP_1)$$

Equivalent formulation(Lagrangian Relaxation) :

$$\min_x \ \lambda\|x\|_1 + \frac{1}{2}\|Ax-b\|_2^2 \qquad (Q_\lambda)$$

LR is easier to solve than P_1 .

Gradient Projection Sparse Reconstruction(GPSR)

LR can be reformulated to

$$\min_{z\geq 0} \frac{1}{2} z^* B z + c^* z \tag{BCQP}$$

with

$$z = \begin{bmatrix} x^+ \\ x^- \end{bmatrix}, c = \begin{bmatrix} \lambda + A^* b \\ \lambda - A^* b \end{bmatrix} \text{ and } B = \begin{bmatrix} A^* A & -A^* A \\ -A^* A & A^* A \end{bmatrix} \tag{28}$$

Efficiently resolved (BCQP) by the general Gradient Projection method.

Direct solution : Linear Programming

P_1 can be transformed into Linear Programming :

$$\min \sum_k (x_k^+ + x_k^-) \ \text{s.t.}\ A(x^+ - x^-) = b, \text{ and } x^+, x^- \geq 0 \quad (29)$$

then solved by :

- Simplex method
- Interior Point method

Generally slow when A is

- large scale
- dense
- submatrix of fast orthonormal transform(DFT, Wavelet).

Indirect solution

General convex optimization problem :

$$\min_x J(x) + H(x) \tag{B_0}$$

- data fitting term : $H(x) = \|Ax - b\|_2^2/2$
- regularization term : $J(x) = \lambda\|x\|_1$

Bregman relaxation :

$$x^* = \arg\min_x B_J^p(x, x') + H(x) \tag{B_1}$$

with *Bregman distance* :

$$B_J^p(u, v) = J(u) - J(v) - \langle p, u - v\rangle\,,\ p \in \partial J(v) \tag{30}$$

Bregman Iterative Regularization

Idea of BIR :

$$x^* \text{ solves } (B_1) \quad \Leftrightarrow \quad 0 \in \partial\{B_J^p(x, x') + H(x)\}$$

which gives :

$$p - \nabla H(x^*) \in \partial J(x^*)$$

Bregman Iterative Regularization :

```
Initialization : k = 0, p_0 = 0, x_0 = 0
while Not converged do
    x_{k+1} ← arg min_x B_J^{p_k}(x, x_k) + H(x)
    p_{k+1} ← p_k − ∇H(x_{k+1}) ∈ ∂J(x_{k+1})
    k ← k + 1
end while
```

Convergence guaranteed in Bregman distance.

Bregman Iterative Regularization

At step k, $\min_x B_J^{p_k}(x, x_k) + \frac{1}{2}\|Ax - b\|_2^2$ is equivalent to :

$$\min_x J(x) + \frac{1}{2}\|Ax - b_{k+1}\|_2^2, \text{ with } b_{k+1} = b + b_k - Ax_k \quad (31)$$

which can be efficiently solved by GPSR.

1: Initialization : $k = 0, b_0 = 0, x_0 = 0$
2: **while** Not converged **do**
3: $\quad b_{k+1} \leftarrow b + b_k - Ax_k$
4: $\quad x_{k+1} \leftarrow \arg\min_x J(x) + \|Ax - b_{k+1}\|_2^2/2$
5: $\quad k \leftarrow k + 1$
6: **end while**

In finite steps, x_k converges to the true solution of P_1 .

Conclusion

Sparse reconstruction works thanks :

- Sparse representation of object in (redundant)dictionary
- Efficient numerical algorithms
- Sensing matrix A well-behaved

Challenges :

- Tomography projector A : Polar Fourier Transform
- Numerical methods non predicted by theoretical analysis
- Bridging the representation and reconstruction problems