Regularization in Banach spaces

Kamil S. Kazimierski

Center for Industrial Mathematics (ZeTeM), Bremen

March 2009

- Bremen = Brême
- Population:
  500 000 - 800 000
- Group Prof. Maaß
- AG Technomathematik = Industrial Mathematics
- 10-15 people
- Mico-Cutting Processes, Impedance Tomography, Image Processing, Inexact Operators, Sparsity, ...

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Ill-posed Problems, e.g. CT, MRT...

$A : X \to Y$ linear, X, Y Banach

$$Ax = y \quad \|y - y^\delta\| \leq \delta$$

Problem: Construct x_δ with $x_\delta \to x$ for $\delta \to 0$

Ill-posed Problems, e.g. CT, MRT...

$A : X \to Y$ linear, X, Y Banach

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Problem: Construct x_δ with $x_\delta \to x$ for $\delta \to 0$
Idea:

$$x_\delta = A^{-1} y_\delta$$

Problems:

- Solution does not exist
- Solution is not unique
- Solution does not depend continuously on y^δ

$\rightsquigarrow$ ill-posed problem $\rightsquigarrow$ need for regularization

Banach spaces - The Good, the Bad and the Ugly

- sequence spaces ℓ^p
- Lebesgue L^p
- Sobolev W^s_p
- Total-Variation TV
- Besov $B^s_{p,q}$

Banach spaces - The Good, the Bad and the Ugly

- sequence spaces ℓ^p
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- Besov $B_{p,q}^s$
- BMO, Orlicz, Hardy...

Why Banach?

- $\ell^1 \rightarrow$ sparsity
- $L^2 \rightarrow$ smoothing
- approximation results $\rightarrow B^s_{p,q}$
- domain spaces with step functions $\rightarrow$ higher smoothness in $B^s_{p,q}$ scale, cmp. to W^s_2 scale
- ...

Nice spaces, not so nice spaces...

- ℓ^1 not reflexive, good to handle with basic tools (calculus on components)
- ℓ^p $1 < p < \infty$ reflexive, Clarkson, Hanner..., ℓ^2 even Hilbert
- ℓ^∞ not reflexive, no current applications?
- *TV* not reflexive, hard to handle on multidimensional domains, even computation of the norm not trivial.
- Hilbert spaces reflexive, inner product... nice geometry (polarization and parallelogram identities)

nice Banach spaces $\leftrightarrow$ nice geometry

Banach space geometry

Definition (Duality mapping $J_p : X \rightrightarrows X^*$ important!)

$$J_p(x) := \partial\{\tfrac{1}{p}\|\cdot\|^p\}(x)$$

In nice spaces

- Derivative

$$J_p = \nabla \tfrac{1}{p}\|\cdot\|^p$$

- Transport

$$J^*_{p'} J_p = I \qquad J_p J^*_{p'} = I^*$$

- In Hilbert spaces $J_2 = I$ (else nonlinear)

Banach space geometry

Definition (p-smoothness, p-convexity)

$$\frac{1}{p}\|x-y\|^p \leq \frac{1}{p}\|x\|^p - \langle J_p(x), y\rangle + \frac{G_p}{p}\|y\|^p$$

$$\frac{1}{p}\|x-y\|^p \geq \frac{1}{p}\|x\|^p - \langle J_p(x), y\rangle + \frac{C_p}{p}\|y\|^p$$

Examples

- $\ell^p, L^p, B^s_{p,r}, W^m_p$ are $\min\{2,p,r\}$-smooth ($1 < p, r < \infty$)
- $\ell^p, L^p, B^s_{p,r}, W^m_p$ are $\max\{2,p,r\}$-convex ($1 < p, r < \infty$)
- Hilbert spaces are 2-smooth and 2-convex (polarization identity)

Case X, Y Hilbert

Standard

$$\Psi(x) = \frac{1}{2}\|Ax - y^\delta\|_Y^2 + \alpha\frac{1}{2}\|x\|_X^2$$

$$x^\dagger = A^*\omega \Longrightarrow \|x_\alpha^\delta - x^\dagger\| \leq C\sqrt{\delta}$$

$$x^\dagger = (A^*A)^\mu\omega,\ \mu < 1 \Longrightarrow \|x_\alpha^\delta - x^\dagger\| \leq C\delta^{\frac{2\mu}{1+2\mu}}$$

Minimization schemes

$$A^*(Ax - y^\delta) + \alpha x = 0$$

$$x_{n+1} = x_n - \mu(A^*(Ax_n - y^\delta) + \alpha x_n)$$

Case X, Y Banach

$$\Psi(x) = \frac{1}{r}\|Ax - y^\delta\|_Y^r + \alpha\frac{1}{q}\|x\|_X^q$$

FACT (HKPS, Hein)

Minimizers of Ψ can be used as regularizing sequence (for $\delta \to 0$).

$$J_p(x^\dagger) = A^*\omega \Longrightarrow \|x_\alpha^\delta - x^\dagger\|^{c_X} \leq C\delta$$

$$J_{s_X}(x^\dagger) = A^* J_{s_Y} A\omega \Longrightarrow \|x_\alpha^\delta - x^\dagger\|^{c_X} \leq C\delta^{\frac{s_X s_Y}{s_X + s_Y - 1}}$$

QUESTION

Minimization scheme?

Case X, Y Banach

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QUESTION

Minimization scheme? $\rightsquigarrow$ Steepest descent

Descent in the primal

$$x_{n+1} = x_n - \mu_n J^*\big(\nabla\Psi(x_n)\big)$$

μ_n via line search

Y r-smooth and X p-smooth and s-convex, then (x_n) converge strongly to the minimizer of $\Psi(x) := \frac{1}{r}\|Ax - y\|_Y^r + \frac{\alpha}{p}\|x\|_X^p$

Descent in the primal

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(Almost) sparse case

$$\Psi(x) := \frac{1}{2}\|Ax - y\|_2^2 + \alpha \frac{1}{1.01}\|x\|_{1.01}^{1.01}$$

Descent in the dual

$$x_{n+1}^* \in x_n^* - \mu_n \partial \Psi(x_n) \qquad x_{n+1} \in J^*(x_{n+1}^*)$$

μ_n small enough

Y arbitrary, X q-convex, then (x_n) converge strongly to the minimizer of $\Psi(x) = \frac{1}{r}\|Ax - y^\delta\|_Y^r + \alpha \frac{1}{q}\|x\|_X^q$

Descent in the dual

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(Almost) sparse case

$$\Psi(x) := \frac{1}{2}\|Ax - y\|_2^2 + \alpha\frac{1}{2}\|x\|_{1.01}^2$$

Convergence rates (dual)

X q-convex

$$\|x_n - x_\alpha^\delta\| \leq C \cdot n^{-\frac{1}{q(q-1)}}$$

X q-convex, Y r-convex, $M = \max\{q, r\} > 2$

$$\|x_n - x_\alpha^\delta\| \leq C \cdot n^{-\frac{(M-1)}{[(M-1)(q-1)-1]q}}$$

X 2-convex, Y 2-convex

$$\|x_n - x_\alpha^\delta\| \leq C \cdot \exp(-n/C)$$

Convergence rates (primal)

X s_X-smooth, c_X-convex Y s_Y-smooth $s = \min\{s_X, s_Y\} < c_X$

$$\|x_n - x_\alpha^\delta\| \leq C \cdot n^{-\frac{s-1}{c_X - s}}$$

X 2-smooth, 2-convex, Y 2-smooth

$$\|x_n - x_\alpha^\delta\| \leq C \cdot \exp(-n/C)$$

Landweber iteration

Standard

$$x_{n+1} = x_n - \mu_n A^*(Ax_n - y^\delta) \ + \text{stoppage crit}$$

$$x^\dagger = A^*\omega \Longrightarrow \|x^\delta_{\alpha(\delta)} - x^\dagger\| \le C\sqrt{\delta}$$

$$x^\dagger = (A^*A)^\mu\omega \Longrightarrow \|x^\delta_{\alpha(\delta)} - x^\dagger\| \le C\delta^{\frac{2\mu}{1+2\mu}}$$

- no saturation
- only one iteration (cmp. to a-posteriori Tik.-reg.)

Iteration in Banach space

Schöpfer, Schuster, Louis

$$J_p(x_{n+1}) = J_p(x_n) - \mu_n A^* J_{r,Y}(Ax_n - y^\delta) \; + \text{stoppage crit}$$

- no convergence rates
- special regularization (only cond. on X)
- usual regularization (cond. on X and Y)
- slow

Iteration in Banach space

K, Hein

$$J_p(x_{n+1}) = J_p(x_n) - \mu A^* J_{r,Y}(Ax_n - y^\delta) + \beta_n J_p(x_n) + \text{stoppage crit}$$

- Convergence rate(s)

$$J_p(x^\dagger) = A^*\omega \Longrightarrow \|x_{n(\delta)} - x^\dagger\| \leq C\sqrt{\delta}$$

- usual regularization (cond. on X)

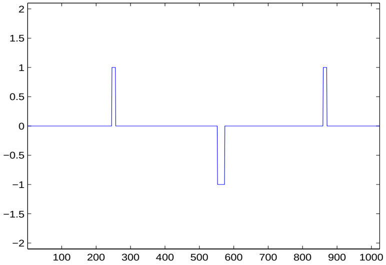
Open problems

- slow algorithms (cmp. to spec. alg.)$\rightarrow$ acceleration?
  RESOP, SESOP
- asymptotic rates $\leftrightarrow$ soph. choice of α, μ_n
- only SC $A^*\omega, A^*JA\omega \rightarrow$ gen. SC?

The end

Thank you for your attention!

An Inverse Problem - Restoring Peaks



$$Ax(t) = \int_0^t x(\tau)d\tau$$

$$\Psi(x) := \frac{1}{2}\|Ax - (Ax + \eta)\|_2^2 + \frac{\alpha}{2}\|x\|_{1.1}^2 \longrightarrow \min$$

An Inverse Problem - Results