Sampling in tomography

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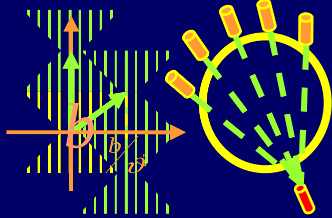


Let $g \in C_0^\infty\left([0,2\pi) \times \mathbf{R}^{n-1}\right)$ be periodic in its first variable

$$\hat{g}_k(\sigma) = \frac{1}{\sqrt{2\pi}^n} \int_0^{2\pi} \int_{\mathbf{R}^{n-1}} g(\phi, s) e^{-i(k\phi + s \bullet \sigma)} ds\, d\phi$$

$$g(\phi, s) = \tilde{\hat{g}}(\phi, s) = \frac{1}{\sqrt{2\pi}^n} \sum_{-\infty}^{+\infty} \int_{\mathbf{R}^{n-1}} \hat{g}_k(\sigma) e^{i(k\phi + s \bullet \sigma)} d\sigma$$

The lattice $L_W = \left\{Wl, l \in Z^n\right\} \cap [0,2\pi) \times R^{n-1}$

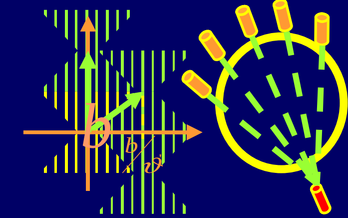
must be a sub-group of $[0,2\pi) \times \mathbf{R}^{n-1}$ (see Faridani 94

(det $W \neq 0$) Faridani and Ritman 00)

If $\mathbf{K} + 2\pi W^{-t} \mathbf{Z}^n$ are disjoints sets **(Shannon conditions)**

$$S_W g(\varphi, s) = \frac{1}{\sqrt{2\pi}^n} \left|\det W\right| \sum_{y \in L_W} f(y) \tilde{\chi}_{\mathbf{K}}\left((\varphi, s) - y\right)$$

$$\left\|S_W g - g\right\|_\infty \leq 2(2\pi)^{-n/2} \sum_{\mathbf{Z} \times \mathbf{R}^{n-1} \setminus \mathbf{K}} \int \left|\hat{g}(\sigma)\right| d\sigma$$

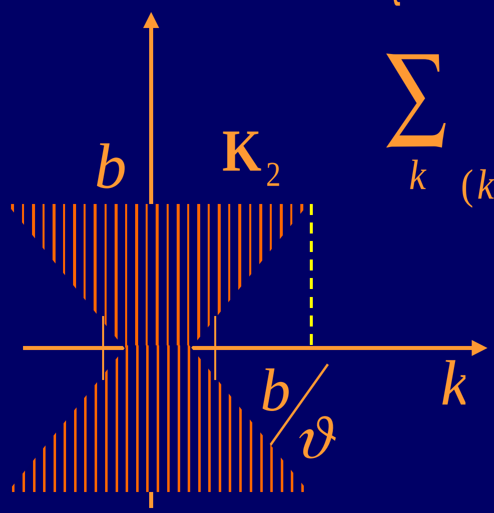


In tomography, we want to sample *g*(ϕ,*s*)=*Rf* (ϕ,*s*)

$$\hat g(\phi,\sigma) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-is\sigma} g(\phi,s)\,ds \qquad \hat g_k(\sigma) = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} e^{-ik\phi}\hat g(\phi,\sigma)\,d\phi$$

If *f* is essentially *b*-band limited, the essential support of is $\hat g_k(\sigma)$

$$\mathbf{K}_2 = \left\{ (k,\sigma) \in Z \times R, |\sigma| < b, |k| < \max\left(\frac{|\sigma|}{\vartheta}, b\left(\frac{1}{\vartheta} - 1\right) \right) \right\}$$

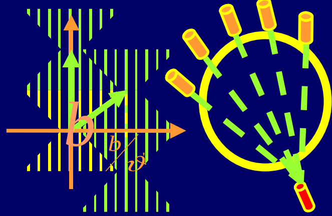


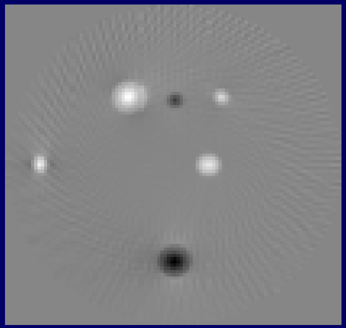
$$\sum_k \int_{(k,\sigma)\notin \mathbf{K}_2} |\hat g_k(\sigma)|\,d\sigma \le \frac{8}{\vartheta\sqrt{2\pi}}\varepsilon_0(f,b) + \eta(\vartheta,b)\|f\|_{L^1}$$

$$0 < \vartheta < 1$$

This term drives the interpolation error

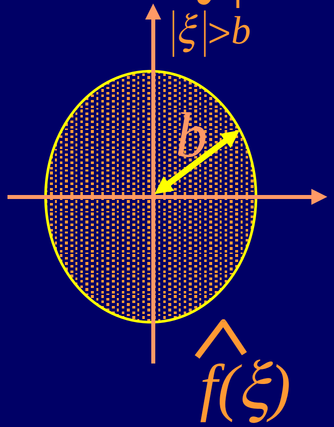
Rattey et Lindgreen 81, Natterer 86



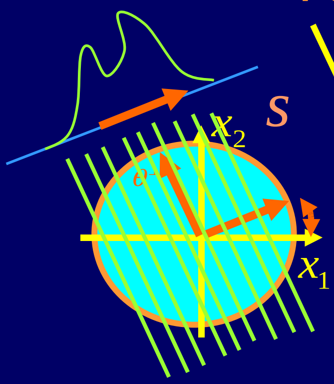


Assumption:

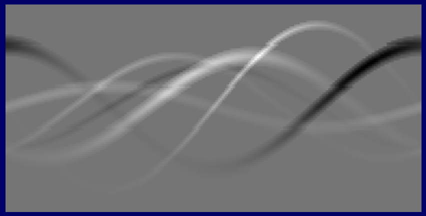
$$\varepsilon_0(f,b) = \int_{|\xi|>b} \left|\hat{f}(\xi)\right| d\xi < \varepsilon$$

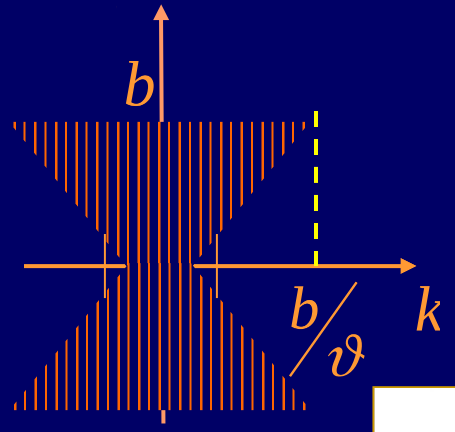


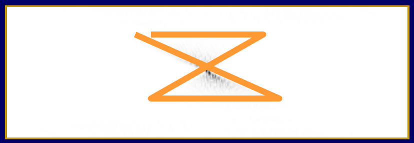
$f(x)$



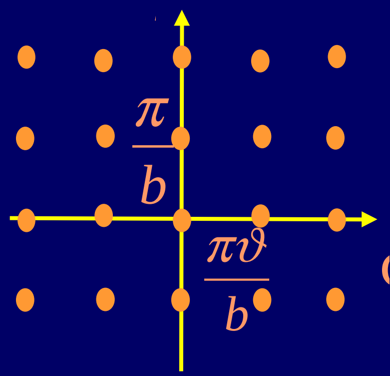
$Rf\,(\phi,s)$



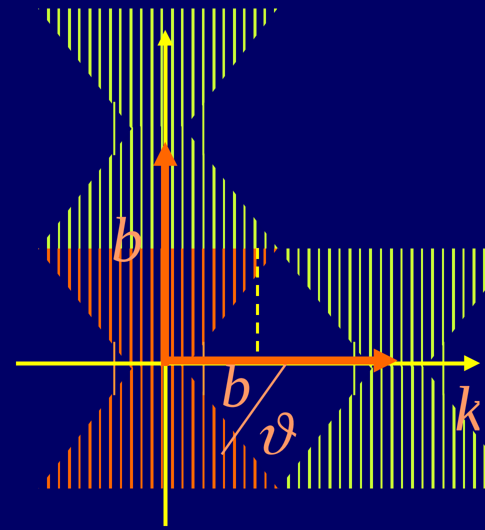




- Standard scheme

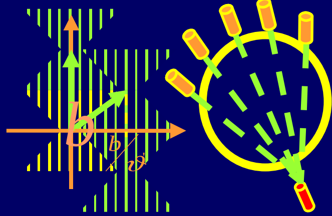


Standard



$$W_S = \frac{\pi}{b}\begin{pmatrix} \vartheta & 0 \\ 0 & 1 \end{pmatrix}$$

$$2\pi W_S^{-t} = 2\begin{pmatrix} b/\vartheta & 0 \\ 0 & b \end{pmatrix}$$



78 Cormack => interlaced sampling

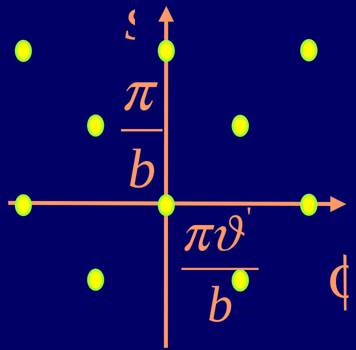
81 Rattey et Lindgren => interlaced sampling and Shannon

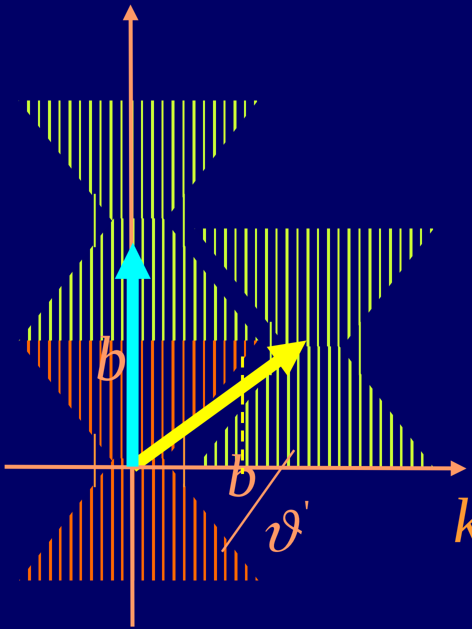
86 Natterer => math. Approach

90,94,00 Faridani => Union of lattices + local tomography

93 Natterer : fan beam sampling conditions

- Interlaced sampling

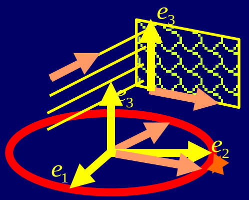




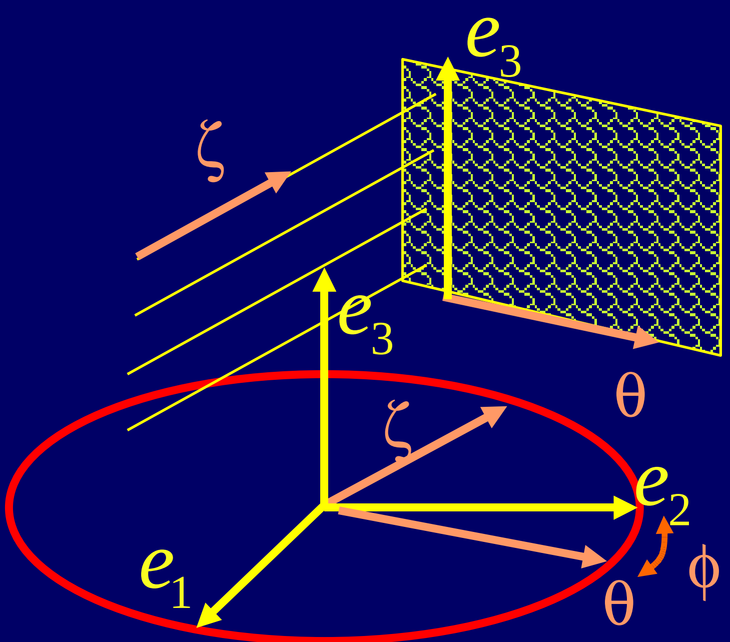
Interlaced

$$W_I = \frac{\pi}{b}\begin{pmatrix} 2\vartheta' & -\vartheta' \\ 0 & 1 \end{pmatrix}$$

$$2\pi W_I^{-t} = \begin{pmatrix} b/\vartheta' & 0 \\ b & 2b \end{pmatrix}$$



- 3D X-ray transform (parallel beam) :



$$Pf(\zeta, x) = \int_{-\infty}^{\infty} f(x + t\zeta)\,dt$$

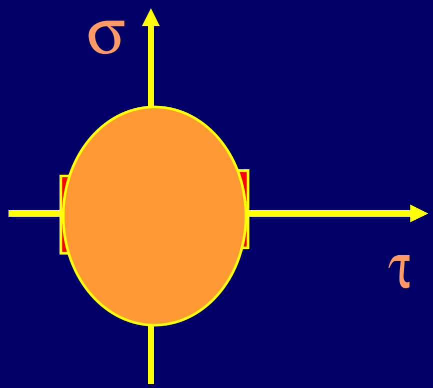
$$\zeta \in \mathbf{S}^2 \Rightarrow \zeta \in \mathbf{S}^1, x \in \theta^{\perp}$$

$$g(\varphi, s, t) = Pf(\zeta, s\theta + te_3) = \int_{-\infty}^{\infty} f(s\theta + te_3 + u\zeta)\,du$$

- Essential support of the Fourier transform of the 3D//XRT of a *b* band limited function

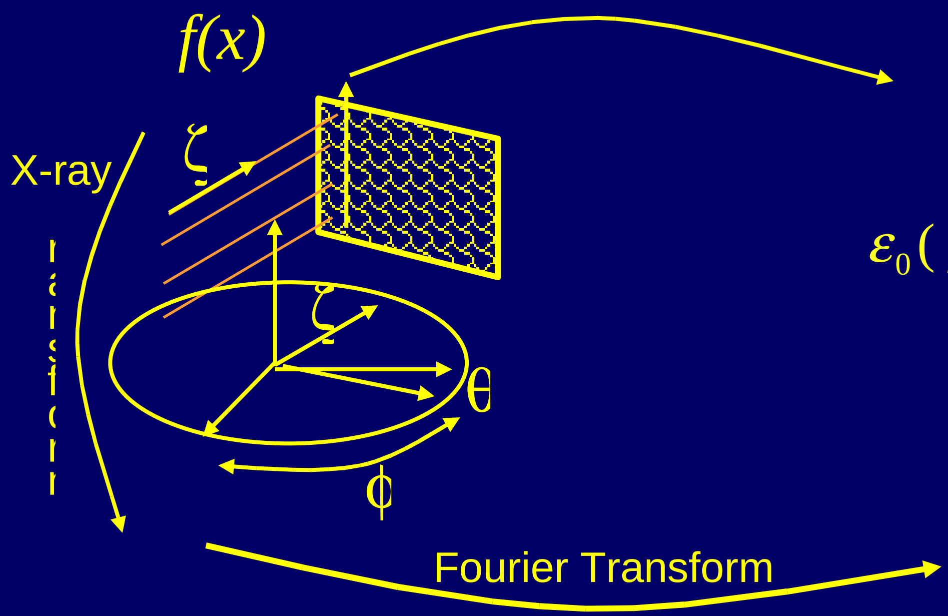
$$\mathbf{K}_3 := \left\{ (k,\sigma,\tau) \in \mathbf{Z}\times\mathbf{R}\times\mathbf{R}, |\sigma| < b, |k| < \max\left(\frac{|\sigma|}{\vartheta}, b\left(\frac{|1|}{\vartheta}-1\right)\right), \tau < c(b,\sigma) \right\}$$

$$c(b,\sigma) = \begin{cases} b & \text{si } |\sigma| < \sigma_{\vartheta,b}\max(1,(1-\vartheta)b) \\ \sqrt{b^2-\sigma^2} & \text{si } \sigma_{\vartheta,b} \le |\sigma| < b \end{cases}$$

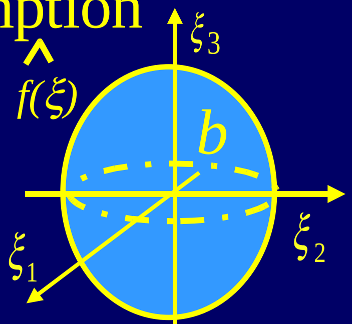


$$\sum_k \int_{(k,\sigma,\tau)\notin \mathbf{K}_3} |\hat{g}_k(\sigma,\tau)|\, d\sigma d\tau \le C_1 \eta(\vartheta,(1/\vartheta-1)b) + C_2 \varepsilon_0(f,b)$$

Sampling conditions

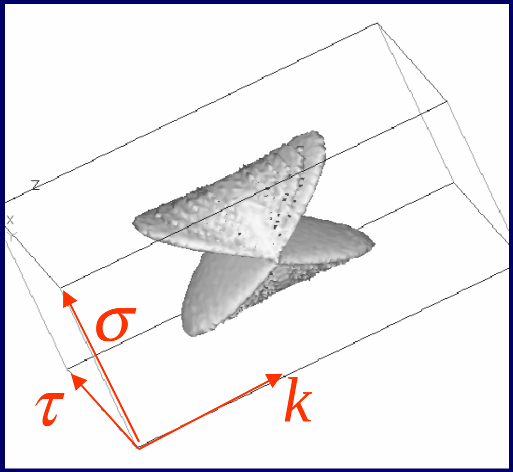


Assumption



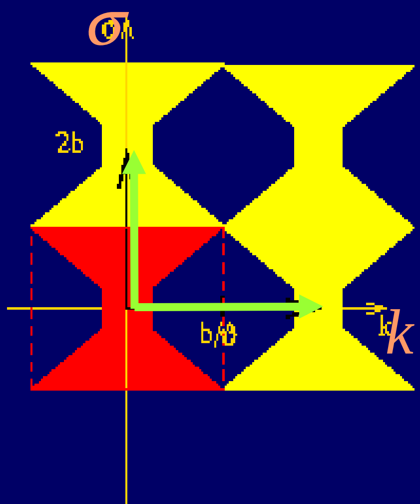
$$\varepsilon_0(f,b) = \int_{|\xi|>b} \left|\hat{f}(\xi)\right| d\xi < \varepsilon$$

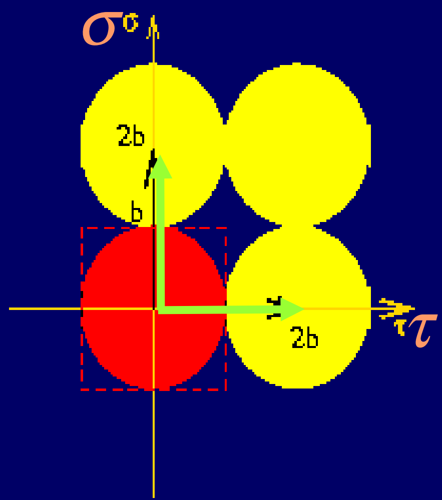
$$g(\varphi, s, t) = \int_{-\infty}^{\infty} f(s\theta + te_3 + u\zeta)\,du$$



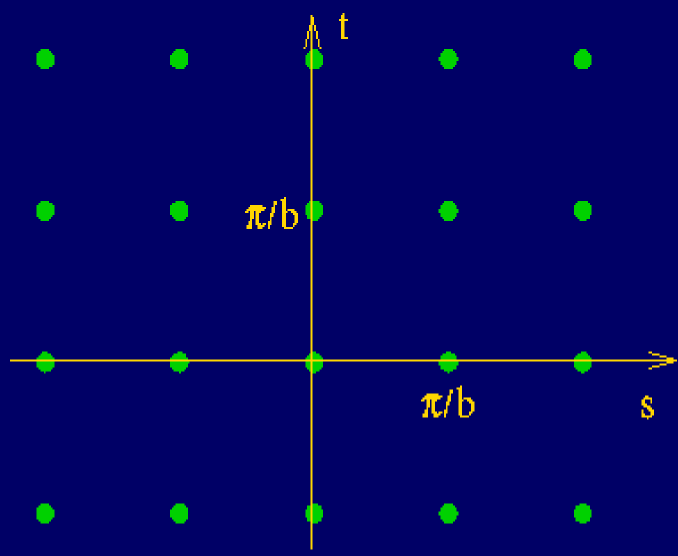
Desbat, 3D95

- Standard Scheme



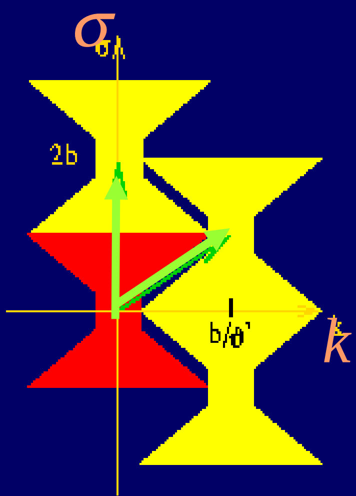


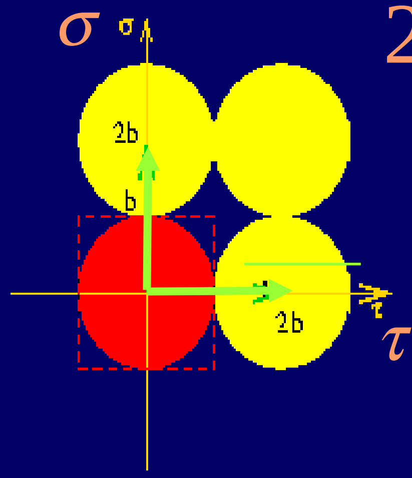
$$2\pi W_S^{-t} = 2b\begin{pmatrix} 1/\vartheta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

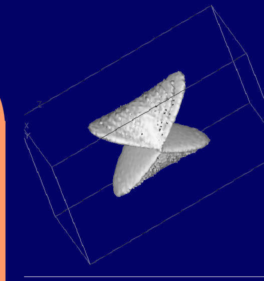


$$W_S = \frac{\pi}{b}\begin{pmatrix} \vartheta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Interlaced scheme



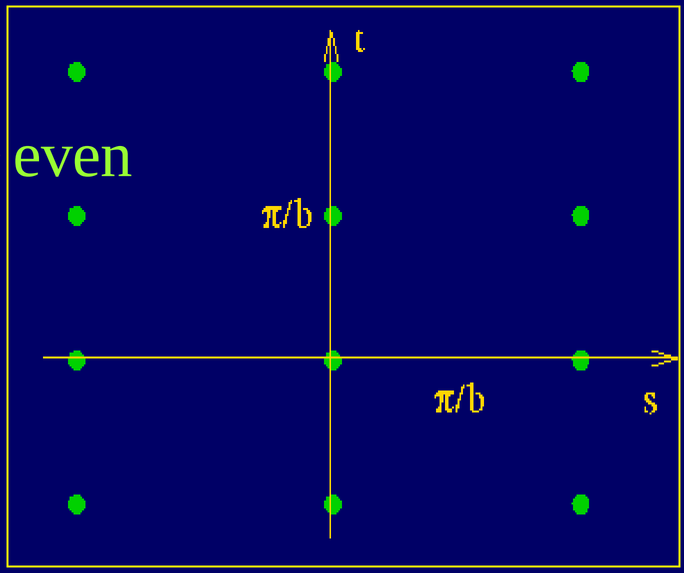


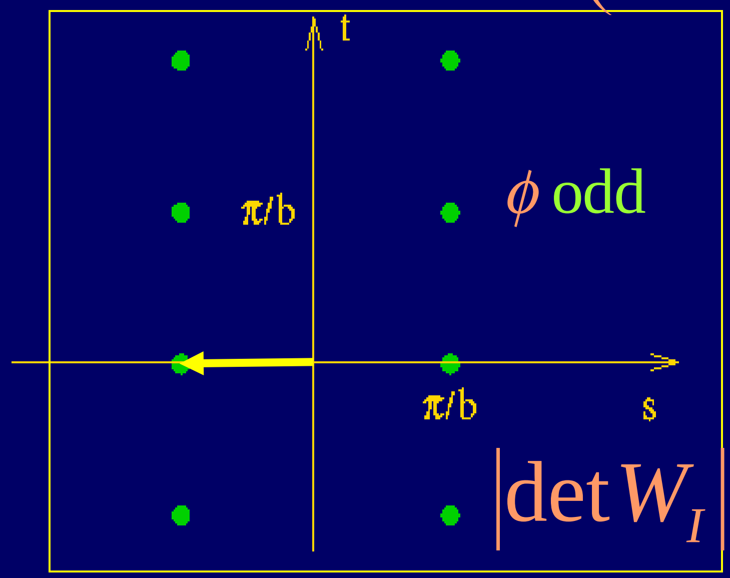


$$2\pi W_I^{-t} = b\begin{pmatrix} 1/\vartheta' & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$W_I = \frac{\pi}{b}\begin{pmatrix} 2\vartheta' & -\vartheta' & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

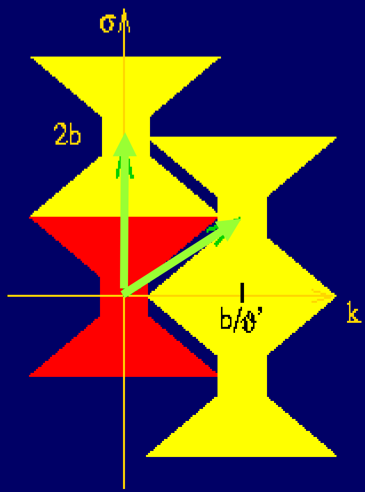
ϕ even

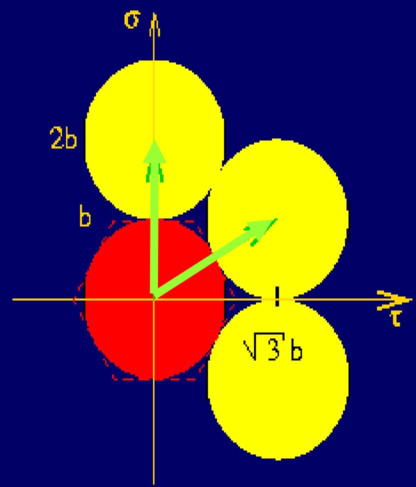


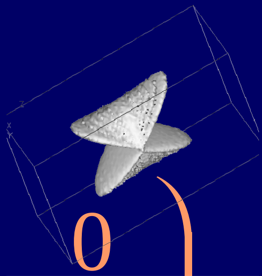


$$\left|\det W_I\right| \approx 2\left|\det W_S\right|$$

- Hexagonal Interlaced scheme

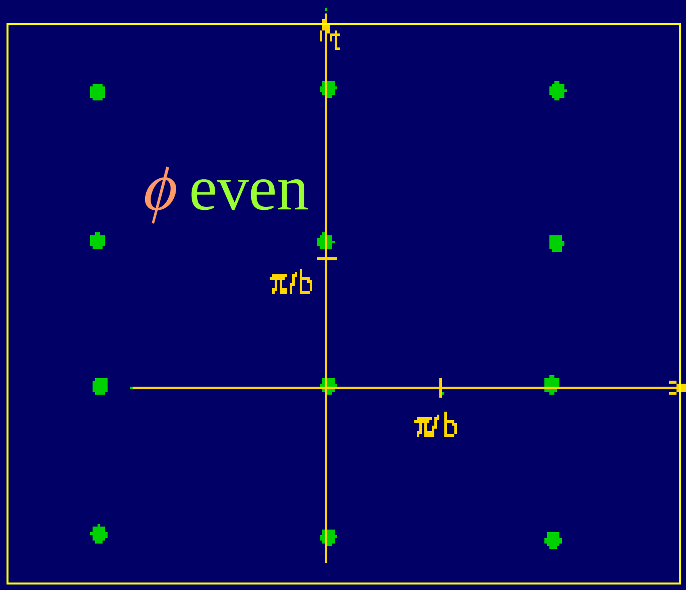


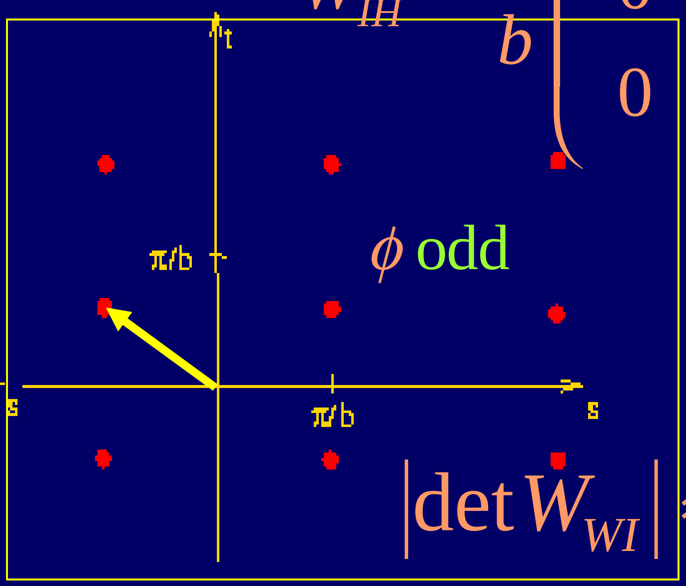




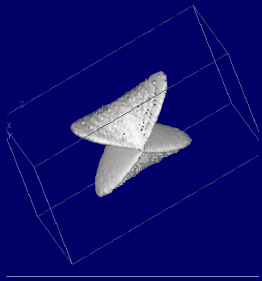
$$2\pi W_{IH}^{-t} = b\begin{pmatrix} 1/\vartheta' & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & \sqrt{3} \end{pmatrix}$$

$$W_{IH} = \frac{\pi}{b}\begin{pmatrix} 2\vartheta' & -\vartheta' & 0 \\ 0 & 1 & 0 \\ 0 & \frac{-1}{\sqrt{3}} & \frac{2}{\sqrt{3}} \end{pmatrix}$$





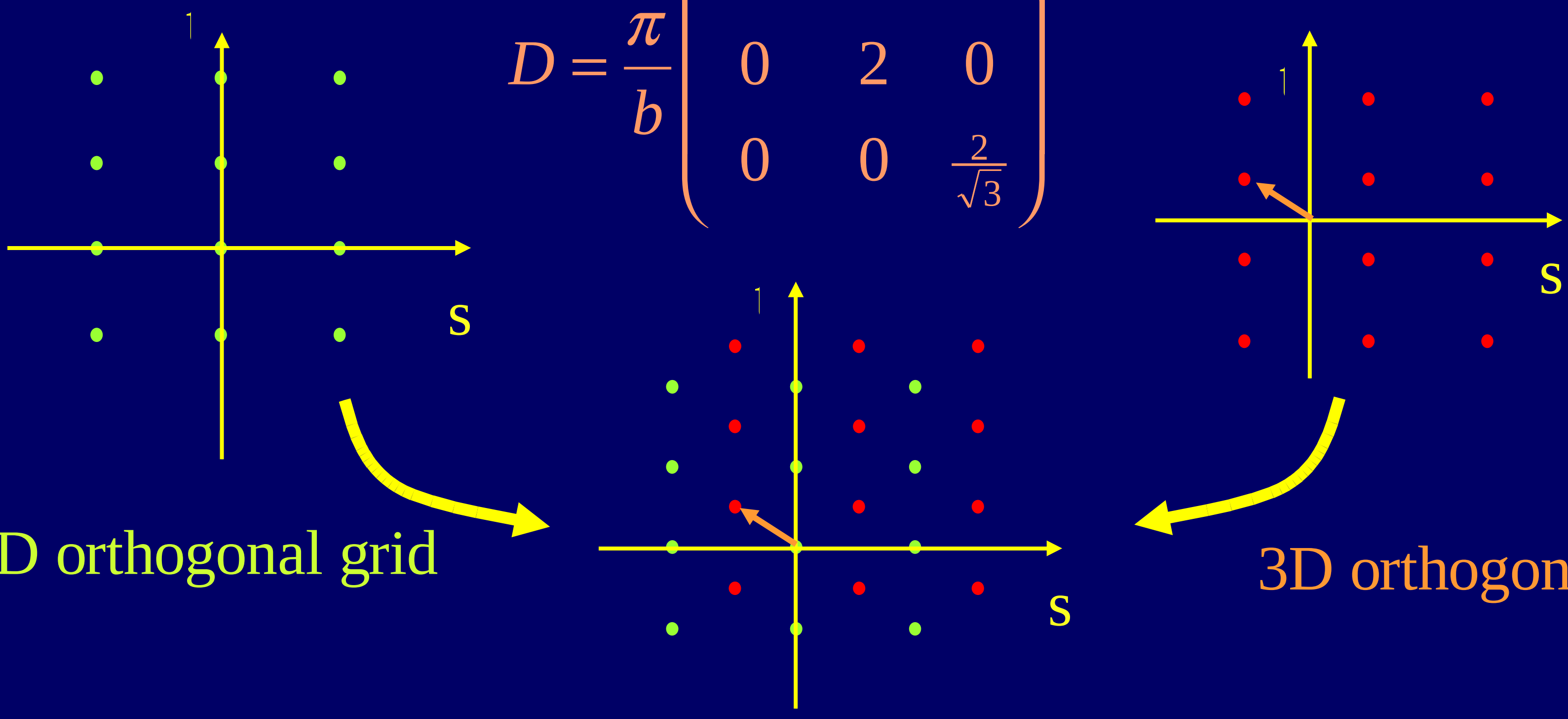
$$\left|\det W_{WI}\right| \approx \frac{4}{\sqrt{3}}\left|\det W_S\right|$$



Angles $2k\phi$

$$D = \frac{\pi}{b}\begin{pmatrix} 2\vartheta' & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{2}{\sqrt{3}} \end{pmatrix}$$

Angles $(2k+1)\phi$

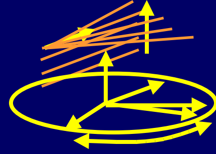


3D orthogonal grid

3D orthogonal grid

Apply Faridani 94 for new efficient schemes

3D Fan Beam X-Ray Transform

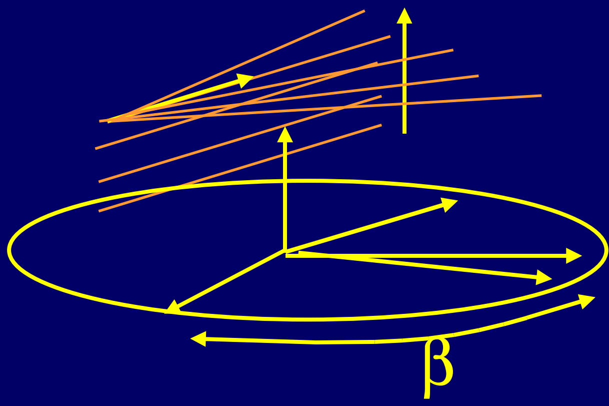


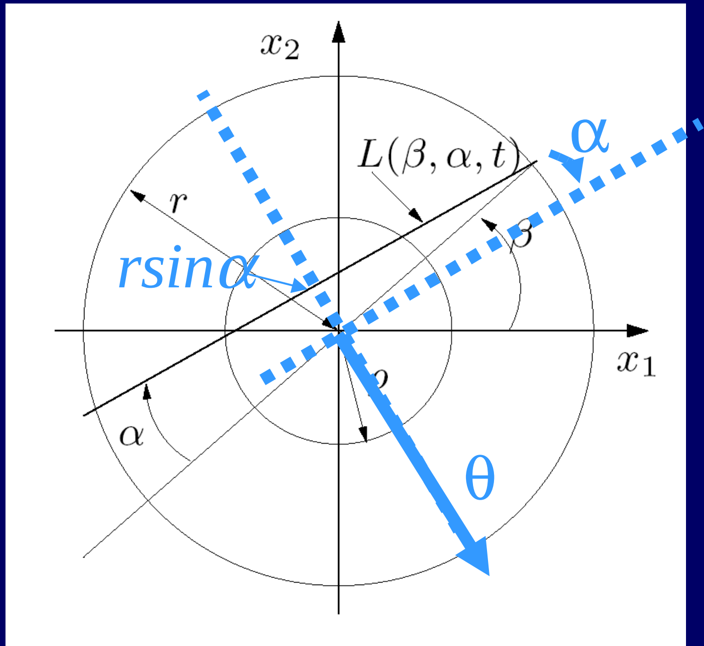
Fan beam geometry at each plane perpendicular to e_3 :

$$g(\beta,\alpha,t) = \mathcal{D}_{e_3^\perp} f(\beta,\alpha,t) = \int_{L_{\beta,\alpha,t}} f(x)dx,$$

Link with the parallel 3D X-ray Transform :

$$\mathcal{D}_{e_3^\perp} f(\beta,\alpha,t) = \mathcal{P} f(\beta+\alpha-\pi/2, r\sin\alpha, t)$$



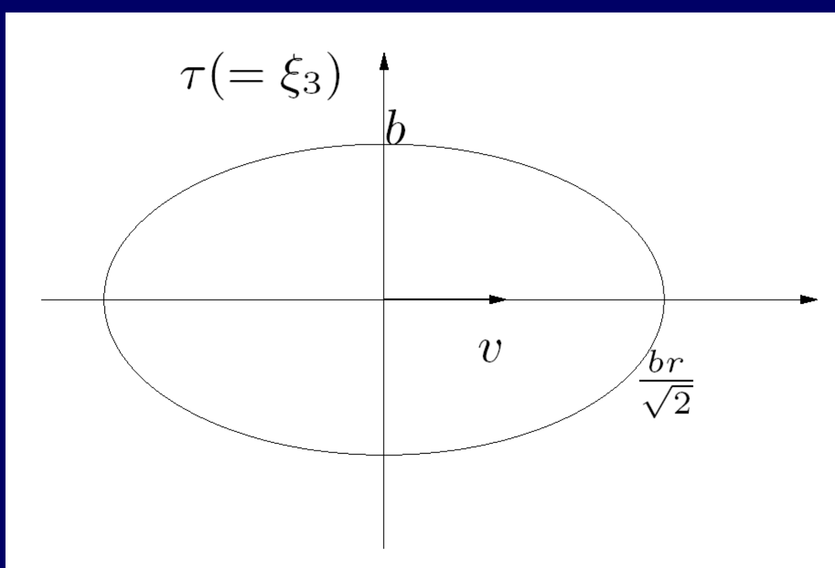


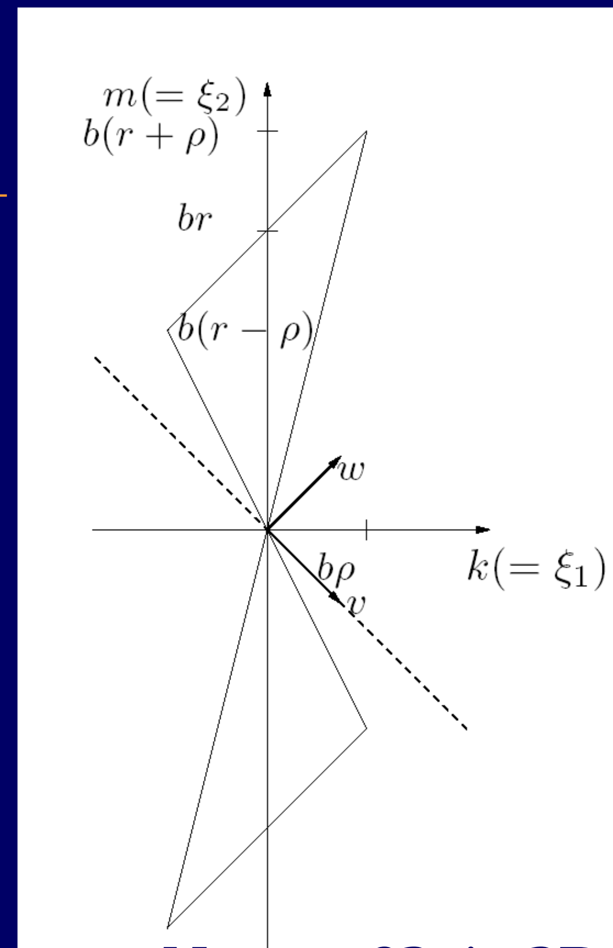
Main result

$$K_{\mathcal{D}_{e_3{}^\perp}} = \{(k, m, \tau) \in \mathbb{Z} \times 2\mathbb{Z} \times \mathbb{R}; |k - m|^2 + r^2\tau^2 < r^2 b^2, |k|r < |k - m|\rho\}$$

$\left|\hat{g}(\xi)\right|$ negligible outside of $K_{D_{e_3^\perp}}$

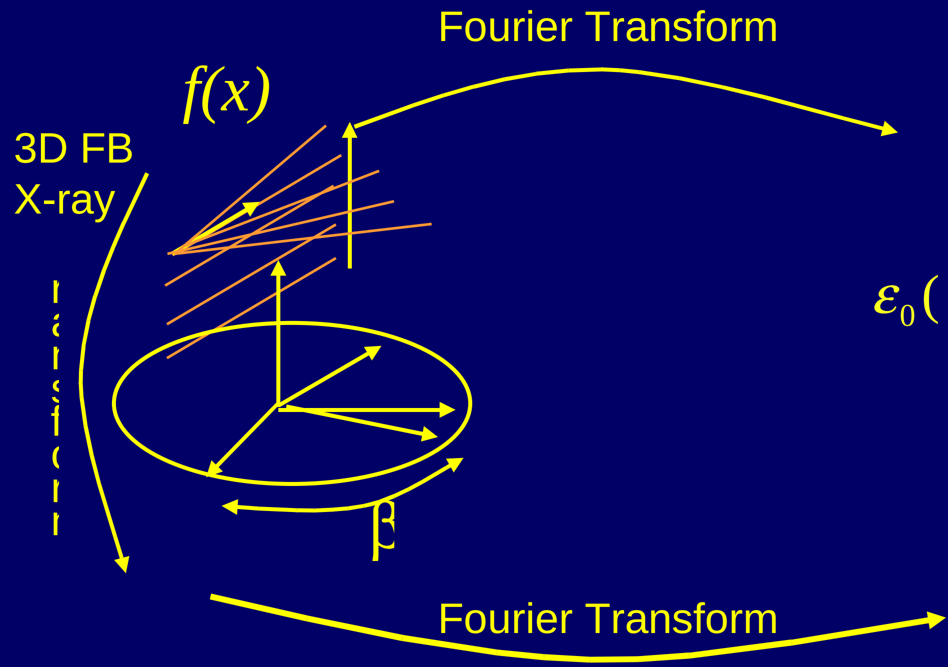
if f is essentially b band-limited

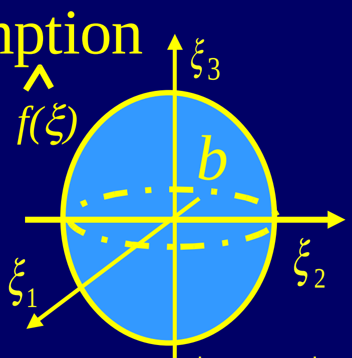




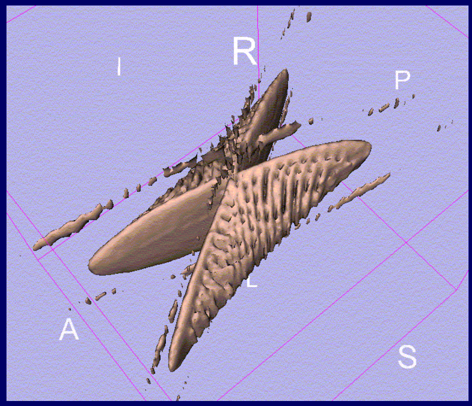
Natterer93, in 2D

3D // Fan Beam sampling conditions

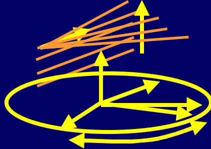




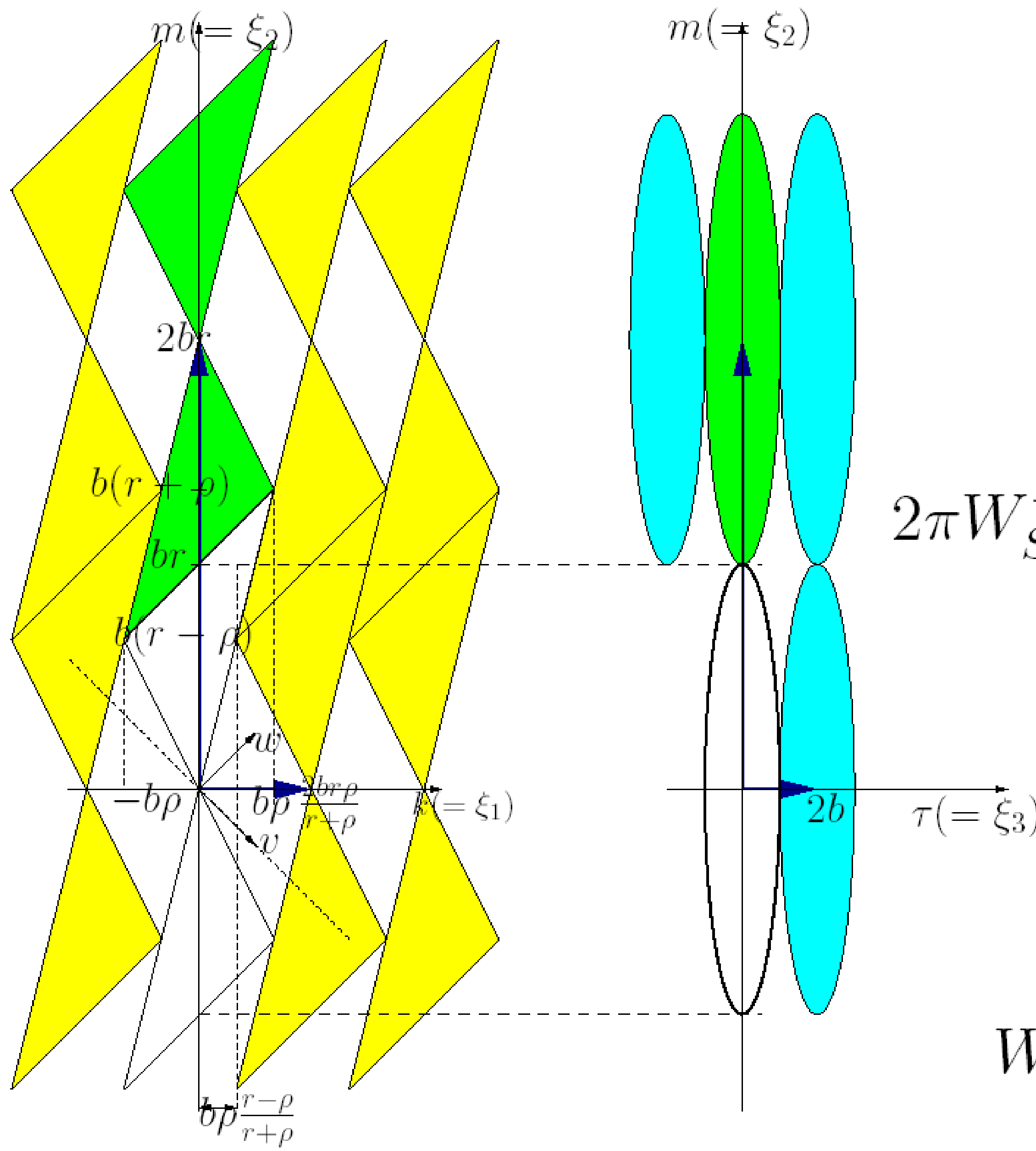
$$\varepsilon_0(f,b) = \int_{|\xi|>b} \left|\hat{f}(\xi)\right| d\xi < \varepsilon$$



$$g(\beta,\alpha,t) = \mathcal{D}_{e_3{}^\perp} f(\beta,\alpha,t) = \int_{L_{\beta,\alpha,t}} f(x)dx,$$



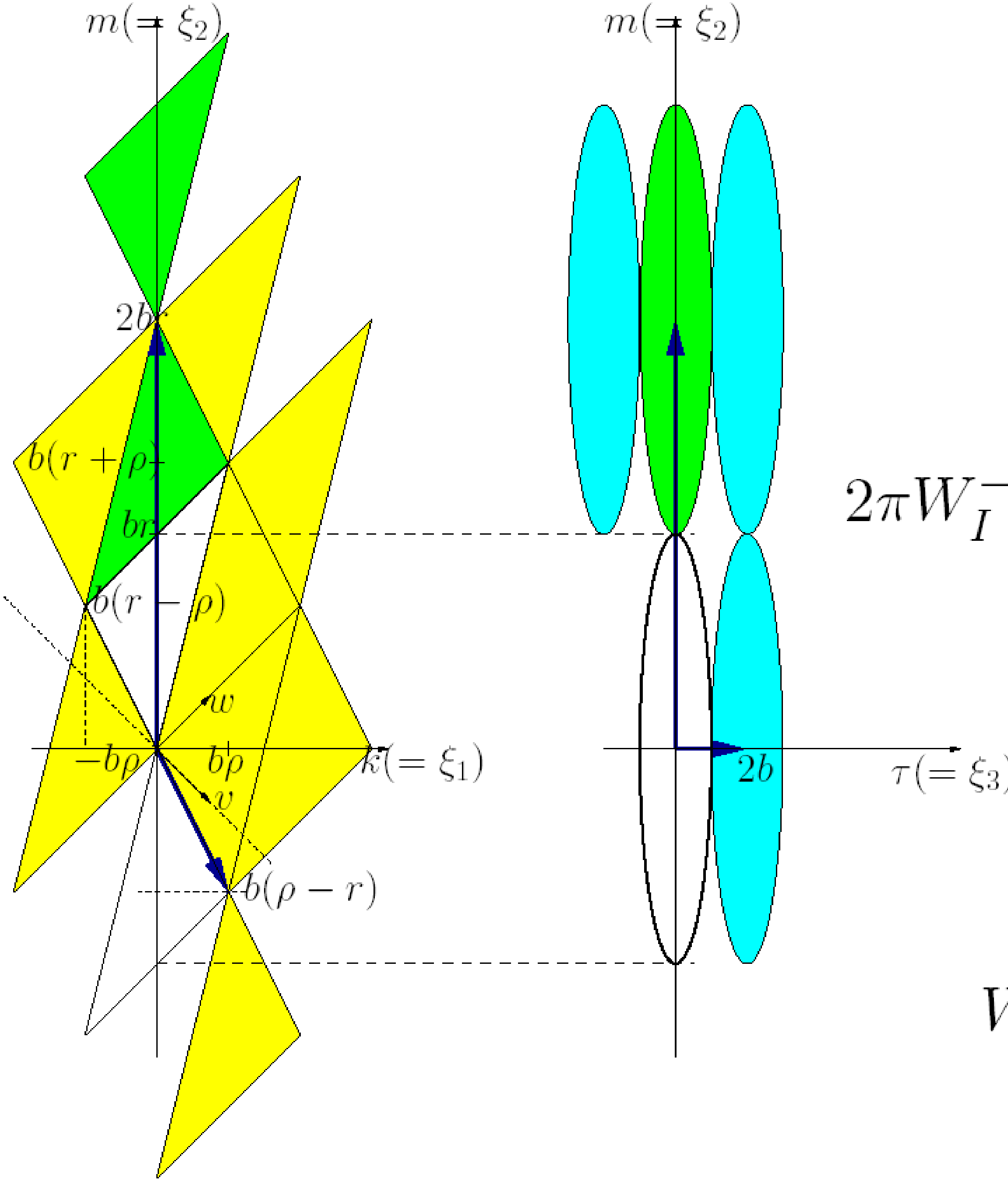
Standard sampling schemes



$$2\pi W_S^{-t} = 2b \begin{bmatrix} \frac{r\rho}{\rho+r} & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$W_S = \frac{\pi}{b} \begin{bmatrix} \frac{\rho+r}{r\rho} & 0 & 0 \\ 0 & \frac{1}{r} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

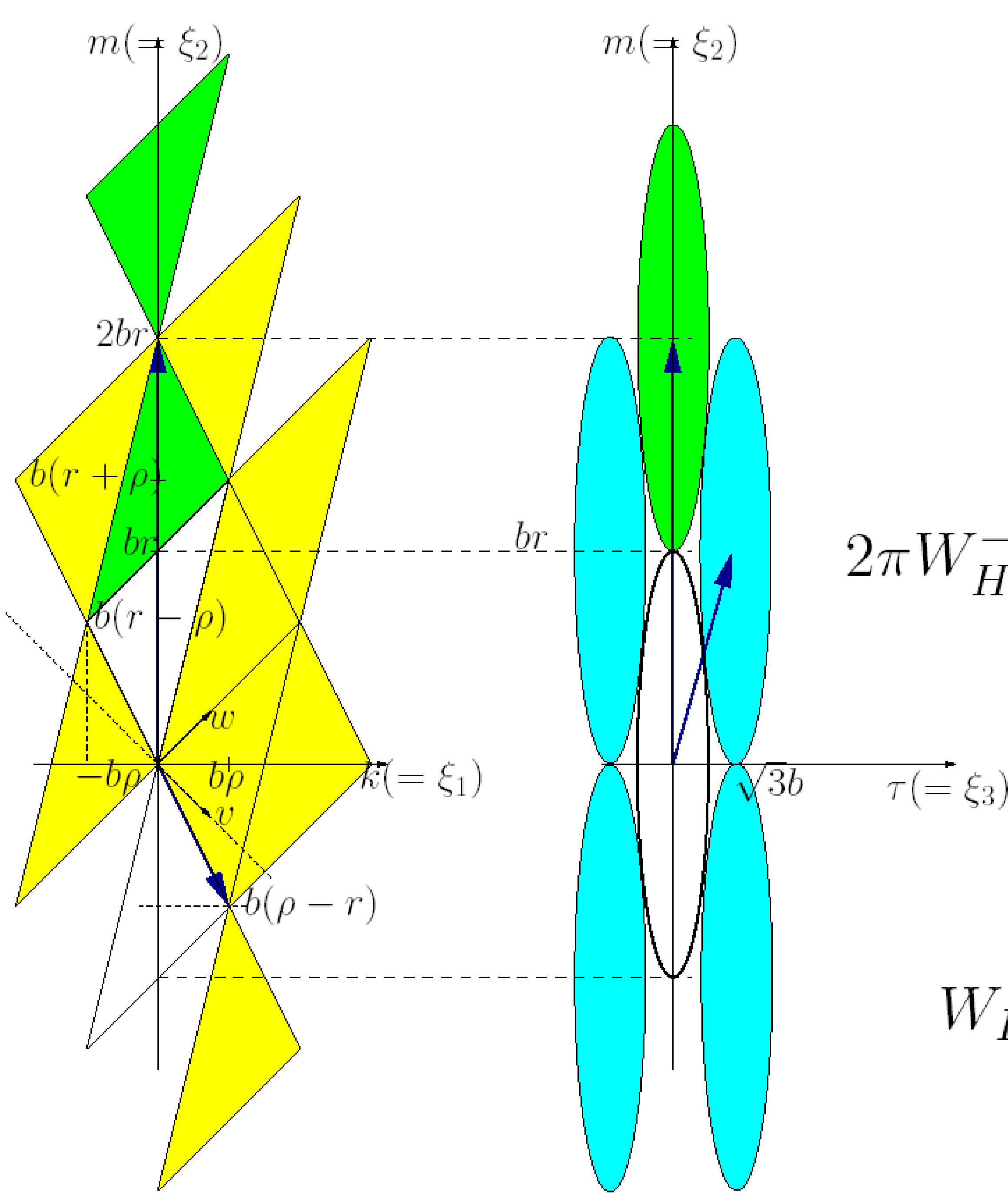
Interlaced sampling schemes



$$2\pi W_I^{-t} = b \begin{bmatrix} \rho & 0 & 0 \\ \rho - r & 2r & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

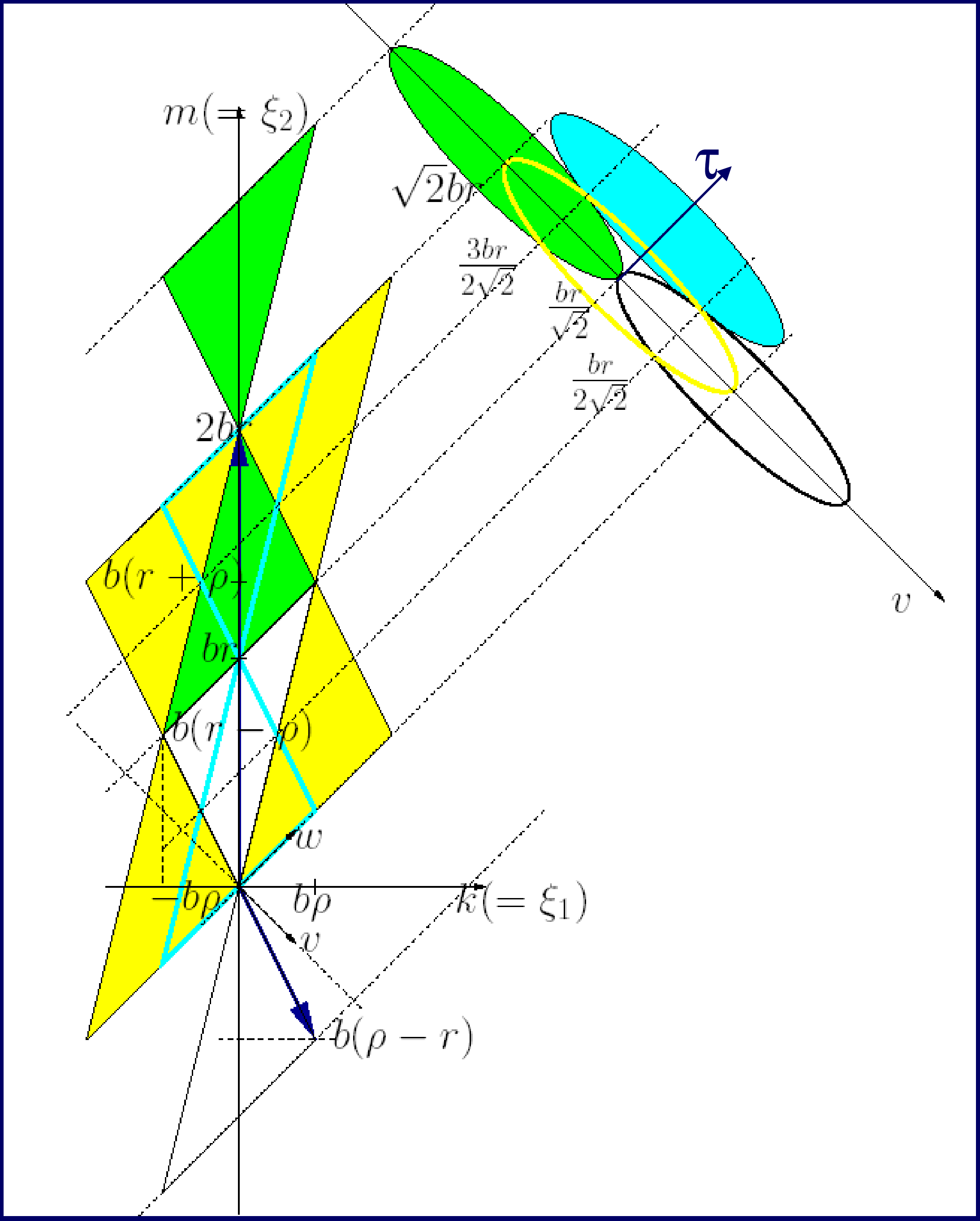
$$W_I = \frac{\pi}{b} \begin{bmatrix} \frac{2}{\rho} & \frac{r-\rho}{r\rho} & 0 \\ 0 & \frac{1}{r} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hexagonal Interlaced sampling schemes



$$2\pi W_{HI}^{-t} = b\begin{bmatrix}\rho & 0 & 0\\ \rho - r & 2r & r\\ 0 & 0 & \sqrt{3}\end{bmatrix}$$

$$W_{HI} = \frac{\pi}{b}\begin{bmatrix}\frac{2}{\rho} & \frac{r-\rho}{r\rho} & 0\\ 0 & \frac{1}{r} & 0\\ 0 & \frac{-1}{\sqrt{3}} & \frac{2}{\sqrt{3}}\end{bmatrix}$$



HI schemes discussion

Scheme efficiency

$$|\det W_{HI}| = \frac{2}{\sqrt{3}} |\det W_I| = \frac{2}{\sqrt{3}} \frac{2\eta}{\eta^2 + \eta} |\det W_S|$$

$$\eta = \frac{\rho}{r} \leq 1$$

$$|\det W_{HI}| > |\det W_I| \geq |\det W_S|$$