Tomographic Reconstruction From Few Projections

A Review on CS Theory and Some Numerical Results

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2 mars 2009

Plan

Formulation of reconstruction problem

Constraint-free formulation :

$$\min_x \frac{1}{2}\|Ax-b\|_2^2 + \lambda\phi(x) \tag{1}$$

Constrainted formulation :

$$\min_x \ \phi(x) \ \text{ s.t. } \ \|Ax-b\|_2^2 \leq \varepsilon \tag{2}$$

Sensing matrix $A \sim PM \times N^2$:

- P : Number of projections
- M : Detector's resolution
- N^2 : Image's resolution

Detector/Image resolution ratio

Numeriqual experience shows :

- $PM \simeq N^2$: Algebraic methods can give nice reconstructions.
- $PM < N^2$: Reconstruction with artifacts.

Few projections : $P << N^2/M$, A is underdetermined.

FIG.: Detector Resolution : 256×16. MSE reconstruction with 16 projections(left) and 8 projections(right)

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Uniqueness of sparse solution

- x is S-sparse if $\|x\|_0 = S$.
- Any $2S$ column of A are independant $\Leftrightarrow$ A is injective on all S-sparse signals.
- S-sparse signal can be uniquely recovered by :

$$\min_x \|x\|_0 \text{ s.t. } Ax = b \qquad (P_0)$$

- Convex relaxation :

$$\min_x \|x\|_1 \text{ s.t. } Ax = b \qquad (P_1)$$

Characterization of sensing matrix

Definition (Restricted Isometry Property)

RIP of A is the smallest $\delta_S \geq 0$ s.t. for all S-sparse signal x :

$$(1 - \delta_S)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \delta_S)\|x\|_2^2 \tag{3}$$

i.e., all S-column of A act like isometry.

- δ_s bounds the singular values of S-submatrix
- Small δ_s, nice behaviour of A
- If $\delta_{2S} = 1$, then some $2S$ submatrix A_{2S} is noninjective.

P_0 and P_1 equivalence

Theorem (Perfect reconstruction)

Given that $2S$-RIP of A is $\delta_{2S} \leq \sqrt{2} - 1$, for a S-sparse true signal x, P_1 solution x^* is **exactly** *x.*

Theorem (Almost Perfect reconstruction)

Under the same hypothesis, for a general true signal x, P_1 solution x^* obeys :

$$\begin{aligned} \|x^* - x\|_1 &\leq C\|x - x_S\|_1 \quad \textit{and} \\ \|x^* - x\|_2 &\leq C\frac{\|x - x_S\|_1}{\sqrt{S}} \end{aligned}$$

where $x_S = (x_1, x_2, ..x_S, 0..)$ is the S biggest term approximation of x.

Robustess of P_1 reconstruction

For noisy observation $b = Ax + \varepsilon$:

$$\min_x \|x\|_1 \text{ s.t. } \|Ax - b\|_2 \leq \varepsilon \qquad (NP_1)$$

Theorem (Robust reconstruction)

Under the same hypothesis, the solution x^* to (8) obeys :

$$\|x^* - x\|_2 \leq C_1 \varepsilon + C_2 \frac{\|x - x_S\|_1}{\sqrt{S}} \qquad (4)$$

Sparsifying Transforms

- Natural objects are sparse under *Sparsifying Transforms* D
- Incorporating the sensing matrix A :

$$\min_\alpha \|\alpha\|_1 \text{ s.t. } AD\alpha = b \text{ or }, \quad (5)$$

$$\min_x \|D^* x\|_1 \text{ s.t. } Ax = b \quad (6)$$

- Final reconstruction :

$$x^* = D\alpha^*, \ \alpha^* \text{ is solution of } (5)$$

Random sensing matrices

Sensing matrices A satisfy RIP with probability $1 - O(e^{-N})$:

- Gaussian and Bernouille matrix : random entries

$$K > CS \log(N/K)$$

- Discret Fourier Transform matrix : random rows

$$K > CS \log N \text{ (conjectured)}$$

- Incoherent sensing and sparsifying transform matrices

K **randomly choosed observations can measure the essential part of** x **!**

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Projected Gradient Method for P_1 Problem

Solve P_1

$$\min \ \|x\|_1 \ \text{ s.t. } \ Ax = b \tag{7}$$

by simple projected gradient algorithm :

- Approximation : $\phi_t(x) = \sqrt{t + x^2} \simeq |x|$
- Projection : $z \to H = Ax = b$

Remark :

- Slow operations
- Not stable wrt sparsifying transform

P_1 Results : Projected Gradient Method 1

FIG.: MSE vs. P_1 . 10 Projections. Detector/Image=16

P_1 Results : Projected Gradient Method 2

FIG.: MSE vs. P_1 . 12 Projections. Detector/Image=16

P_1 Results : Projected Gradient Method 3

FIG.: MSE vs. P_1 . 45 Projections. Detector/Image=4

IAS Method for NP_1 Problem

$$\min_x \ \frac{1}{2}\|Ax - b\|^2 + \mu\|x\|_1 \tag{8}$$

Transformed to BCQP :

$$\min_{z \geq 0} \frac{1}{2} z^* B z + c^* z \qquad \text{(BCQP)}$$

Infeasible Active Set(IAS) method :

- Works on KKT system
- Finite steps convergence(less than 10 usually)
- B must be definite positive
- Sensitive to regularization parameters

NP_1 Results : IAS Method 1

FIG.: MSE vs. P_1 . 12 Projections. Detector/Image=16. Source Intensity=10^5

NP_1 Results : IAS Method 2

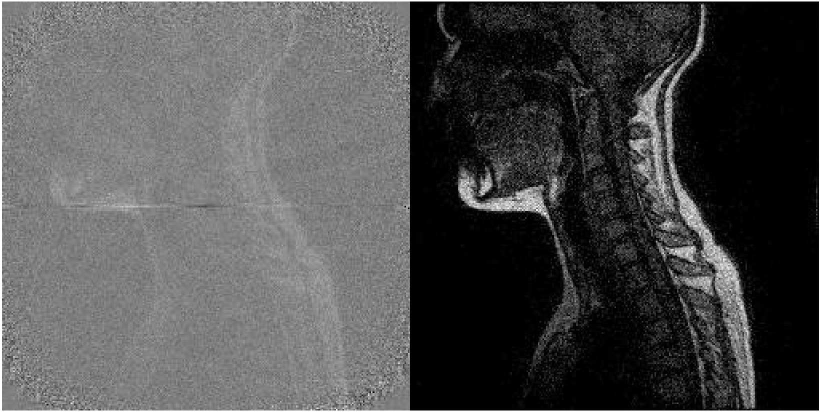

FIG.: MSE vs. P_1 . 12 Projections. Detector/Image=16. Source Intensity=10^4

Wavelet Transform

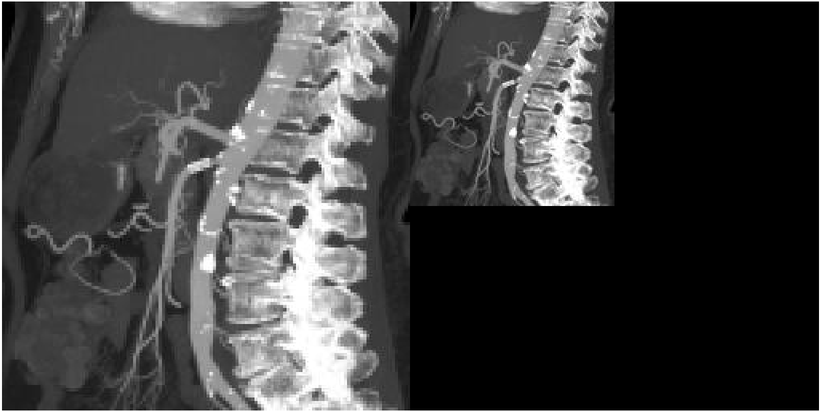

FIG.: Nonsparse image and its wavelet coefficient(sparsity 75%)

Wavelet Transform

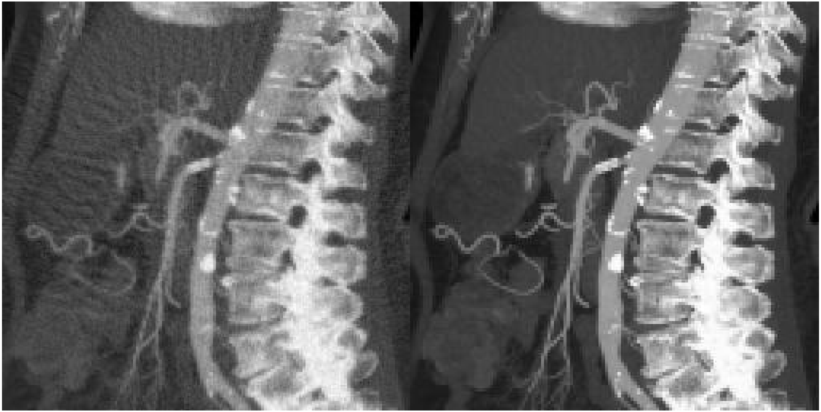

FIG.: MSE(Left) vs haar(Right). Reconstruction with 45 projections. Detector/Image ratio=4.