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Stability estimates for the inversion of the truncated Hilbert transform

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- Context: limited data reconstruction via the differentiated backprojection (DBP).
- Background: asymptotic behaviour of the SVD of the truncated Hilbert transform.
- Stability estimates for the inversion.

Limited data reconstruction via the differentiated backprojection (DBP)

Consider the 2D Radon transform of a smooth function f, with the usual parallel beam parameterization

$$g(s,\phi) = (\mathcal{R}f)(s,\phi) = \int dl \ f(s\, u^{\perp}(\phi) + l\, u(\phi)) \quad ; \quad u(\phi) = (-\sin\phi, \cos\phi)$$

and the backprojection of its radial derivative over a 180 degree interval:

$$b(\mathbf{x} = (x,y)) = \int_0^{\pi} d\phi \left\{ \frac{\partial g(s,\phi)}{\partial s} \right\}_{s=\mathbf{x}\cdot u^{\perp}(\phi)}$$

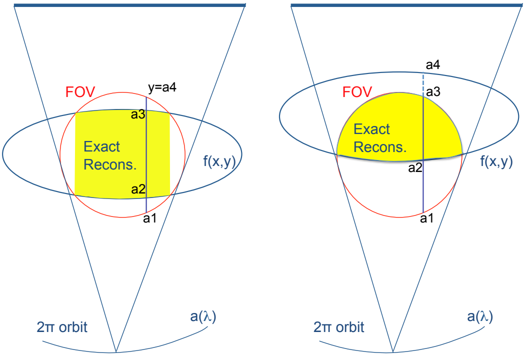
Then one has the fundamental DBP relation

$$b(x,y) = (H_{1_y} f)(x,y) = \frac{1}{\pi} p.v. \int_{-\infty}^{\infty} dy' \ \frac{f(x,y')}{y-y'}$$

$$b(\mathbf{x} = (x, y)) = \int_0^{\pi} d\phi \left\{ \frac{\partial g(s, \phi)}{\partial s} \right\}_{s = \mathbf{x} \cdot u^{\perp}(\phi)}$$

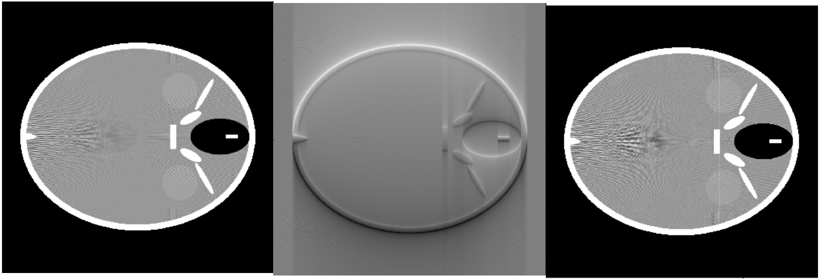
$$b(x, y) = (H_{1_y} f)(x, y) = \frac{1}{\pi} p.v. \int_{-\infty}^{\infty} dy' \, \frac{f(x, y')}{y - y'}$$

- Similar results for fan-beam data and for cone-beam data.
- Reduces 2D or 3D problem to a family of 1D inversions of the Hilbert transform.
- Gelfand and Graev 1991, Finch 2002, Zou et al 2004, Noo et al 2005.
- Allows accurate reconstruction from limited tomographic data sets.
- Generalization: DBP over interval less than 180 degrees $\rightarrow$ sum of two Hilbert transforms.



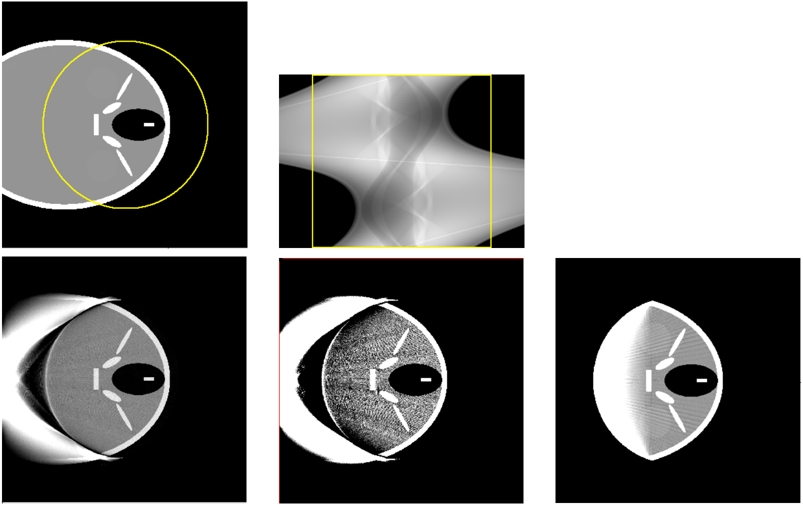
The limited data allow calculating the DBP within FOV.

- Left: $supp(f) = (a_2, a_3) \subset FOV = (a_1, a_4) \Rightarrow$ closed form inversion of the *finite Hilbert transform* (Tricomi etc..)
- Right: $supp(f) = (a_2, a_4)$ overlaps with the $FOV = (a_1, a_3)$: no closed form inversion but uniqueness and stability within part of the FOV (Defrise, Noo, Clackdoyle,Kudo 2006), *Truncated Hilbert transform*.



Example with full noise-free data. Left: FBP reconstruction. Center: the DBP. Right: inverse finite Hilbert transform along each line $x = \text{cst}$. Grey scale: $0.9, 1.15$.

- The stability for noise is similar to FBP: same backprojection, and filters have same behaviour : $|\nu| = |i\nu|$.



Truncated data. Top left: Shifted phantom and FOV. Top right: fan-beam sinogram and FOV. Bottom: SART reconstruction, 200 iterations, $\alpha = 0.25$. Left: scale $(0, 2)$. Center: scale $(0.9, 1.15)$. Right: inverse finite Hilbert transform along each line $x = \text{cst}$. Grey scale: $0.9, 1.15$

Why is the SVD important ?

The generalized solution of a linear inverse problem $g = Hf$ is

$$f^{\dagger} = \sum_n \frac{< g_n, g >}{\sigma_n} f_n$$

with g_n, f_n, σ_n the SVD of the operator (or matrix) H:

$$\begin{aligned} Hf_n &= \sigma_n g_n \\ H^* g_n &= \sigma_n f_n \end{aligned}$$

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SVD of the finite Hilbert transform ($a_1 = a_2 = -1, a_3 = a_4 = 1$).

!! From now on, f(x) is a 1D function !!

Consider $H_F : L^2_w(-1,1) \to L^2_w(-1,1)$, with weighted norm $||f||^2 = \int_{-1}^{1} dy |f(y)|^2 w(y)$ with $w(y) = 1/\sqrt{1-y^2}$:

$$(H_F f)(x) = \frac{1}{\pi} p.v. \int_{-1}^{1} \frac{f(y)dy}{y-x} \quad -1 \leq x \leq 1 \tag{1}$$

The singular system is well-known (Tricomi):

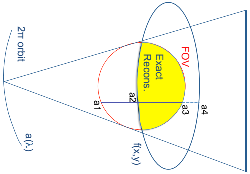
- $\sigma_n = 1, \; n = 0, 1, 2, \cdots$
- $f_n(x) = \sqrt{2/\pi}\,\sqrt{1-y^2}\; U_n(x)$
- $g_n(x) = -\sqrt{2/\pi}\; T_{n+1}(x)$
- $H_F f_n = \sigma_n g_n, \; H_F^* g_n = \sigma_n f_n, \; n = 0, 1, 2, \cdots$, and $H_F^* T_0 = 0$

with U_n, T_n the Chebyshev polynomials: $T_n(x) = \cos n\theta$; $U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}$ with $x = \cos\theta$

Asymptotic of the SVD of the truncated Hilbert transform.

Consider the truncated Hilbert problem for $H_T : L^2(\mathcal{F}) \to L^2(\mathcal{G})$, with $\mathcal{F} = (a_2, a_4)$, $\mathcal{G} = (a_1, a_3)$, and $a_1 < a_2 < a_3 < a_4$:

$$(H_T f)(x) = \frac{1}{\pi} p.v. \int_{a_2}^{a_4} \frac{f(y)dy}{y - x} \quad a_1 \leq x \leq a_3$$



SVD: $g_n \in L^2(a_1, a_3)$, $f_n \in L^2(a_2, a_4)$ and $H_T f_n = \sigma_n g_n$ for $n \in Z$, $\sigma_n \geq \sigma_{n+1}$.

- Spectral properties: Al-Aifari and Katsevich (SIAM Math Anal 2014)
- Asymptotic (for $n \to \pm\infty$): Al-Aifari, MD, Katsevich submitted.
- Technique used by Katsevich (Inv Prob 2010, 2011) and Katsevich and Tobvis (Inv Prob 2012) for the *interior Hilbert transform* where $\mathcal{G} \subset \mathcal{F}$.

Main tools

- A 2nd order differential operator L which commutes with H_T and hence has the same singular functions.
- The characterization of the behaviour of the solutions of $Lf = \lambda f, \ \lambda \in C$ for $x \to a_j^{\pm}$
- The WKB asymptotic form of these solutions for large $|\lambda|$, and the asyptotic form as $x \to a_j^{\pm}$.
- Characterization of the behaviour of f_n and g_n for $x \to a_j^{\pm}$
- Enforcing these conditions restricts the values of λ to a countable set.

Similar approaches used for the Slepian-Pollack problem (extrapolation of band-limited signals, 1960's), the limited-angle Radon transform (Davison and Grunbaum 1980), and the finite Laplace transform (Bertero, Grunbaum, Rebolla 1986).

The differential operator,

$$L(x, d_x)\psi(x) := (P(x)\psi'(x))' + 2(x-\mu)^2\psi(x)$$

where $P(x) = \prod_{j=1}^{4}(x - a_j)$ and $\mu = \frac{1}{4}\sum_{j=1}^{4} a_j$ commutes with H_T.

$\Rightarrow H_T^* H_T$ and L have the same singular functions.

• a_1, a_2, a_3, a_4 are regular singular points $\Rightarrow$ for any $\lambda \in C$ the solutions to $(L-\lambda)\psi = 0$ in a neighborhood of a_i^+ or a_i^- are linear combinations of

$$\psi_1(x) = \sum_{j=0}^{\infty} b_j (x - a_i)^j$$

$$\psi_2(x) = \sum_{j=0}^{\infty} d_j (x - a_i)^j + \ln|x - a_i|\psi_1(x)$$

What can be said of the behaviour of the singular functions $f_n \in L^2(a_2, a_4)$ and $g_n \in L^2(a_1, a_3)$ of H_T ?

First note that if f_n is bounded at a_2, then

$$\begin{aligned} (H_T f_n)(x) &= \frac{1}{\pi} p.v. \int_{a_2}^{a_4} \frac{f_n(y)dy}{y-x} \\ &= \frac{1}{\pi} p.v. \int_{a_2}^{a_4} \frac{(f_n(y) - f_n(a_2))dy}{y-x} + f_n(a_2)\frac{1}{\pi} \underbrace{\log \frac{|a_4 - x|}{|x - a_2|}} \end{aligned}$$

has a log singularity at $x = a_2$.

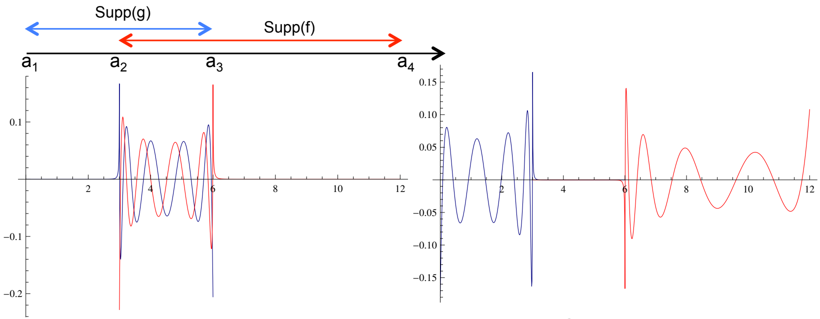
What can be said of the behaviour of the singular functions $f_n \in L^2(a_2, a_4)$ and $g_n \in L^2(a_1, a_3)$ of H_T ?

- $f_n(x) = (1/\sigma_n)(H_T^* g_n)(x)$ is analytic outside $(a_1, a_3) \Rightarrow f_n(x)$ bounded in a_4.
- $f_n(x)$ is bounded at a_2. Indeed if it had a log singularity at a_2^+ then $g_n(x) = (1/\sigma_n)(H_T f_n)(x)$ would not be bounded or have a log singularity there.
- $f_n(x)$ being bounded at a_2, $g_n(x) = (1/\sigma_n)(H_T f_n)(x)$ has a log singularity there.
- the log singularities of g_n at a_2^+ and at a_2^- must be matched otherwise $f_n(x) = (1/\sigma_n)(H_T^* g_n)(x)$ could not be bounded there

$$\Rightarrow \quad \text{close to } a_2 \quad g_n(x) = g_{n,1}(x) + g_{n,2}(x) \log |x - a_2|$$

with $g_{n,1}(x)$ and $g_{n,2}(x)$ continuous.

What can be said of the behaviour of the singular functions $f_n \in L^2(a_2, a_4)$ and $g_n \in L^2(a_1, a_3)$ of H_T ?



Large $\sigma_n \simeq 1$

Small $\sigma_n \simeq> 0$

Example for $a_1 = 0, a_2 = 3, a_3 = 6, a_4 = 12$. Blue: g_n . Red: f_n.

Summary of the properties of f_n and g_n

- $f_n(x)$ bounded in a_2 and a_4, and log singularity in a_3.
- $g_n(x)$ bounded in a_1 and a_3, and log singularity in a_2
- the log singularity of g_n must be matched on a_2^+ and a_2^-, idem for f_n at a_3.
- f_n and g_n are solutions of $L\phi = \lambda\phi$

$\Rightarrow$ Enforcing these conditions for λ large leads to a quantization

$$\sqrt{\lambda_n} = \frac{n\pi}{K_-} + O(n^{-1/2+\delta}) \quad n = 1, 2, 3, \cdots$$

$\Rightarrow$ Asymptotic of the singular values obtained as (essentially) $||H_T f_n||/||f_n||$:

$$\sigma_n = 2e^{-n\pi K_+/K_-}(1 + O(n^{-1/2+\delta})), \; n \to \infty.$$

with $K_- := \int_{a_1}^{a_2} \frac{1}{\sqrt{-P(x)}} dx$ and $K_+ := \int_{a_1}^{a_2} \frac{1}{\sqrt{-P(x)}} dx$.

- Similar derivation for the asymptotic $\sigma_{-n} \to 1$.

Details on the quantification of λ

- The WKB approximation of the solutions for large $|\lambda|$ of the eigenequation

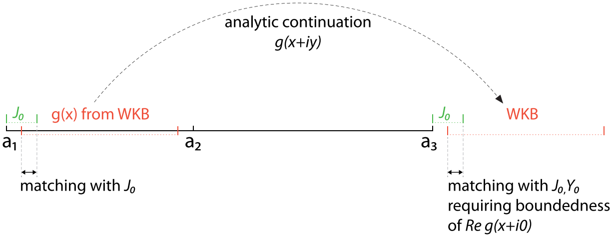
$$(L\phi)(x) - \lambda\ \phi(x) = (P(x)\phi'(x))' + 2(x-\mu)^2\phi(x) - \lambda\Phi(x) = 0$$

where $P(x) = \prod_{j=1}^{4}(x-a_j)$ and $\mu = \frac{1}{4}\sum_{j=1}^{4} a_j$ are linear combinations of

$$\hat{\phi}_\pm(z) \quad = \quad \frac{1}{P(z)^{1/4}} e^{\pm\sqrt{\lambda}\int_{a_1}^{z} \frac{d\xi}{\sqrt{P(\xi)}}} \left(1 + O(|\lambda|^{-\eta_1/2})\right)$$

with uniform accuracy in a region of C excluding neighborhoods of the a_j.

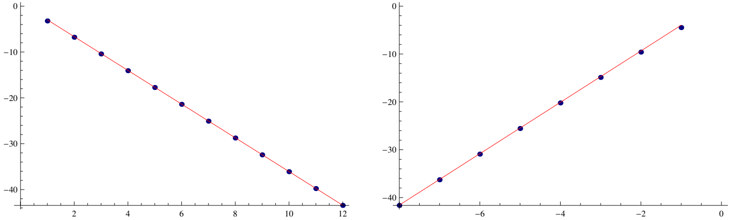
- Close to the a_j the solutions are given in terms of Bessel functions.



Results: singular values

$$\sigma_n = 2e^{-n\pi K_+/K_-}(1 + O(n^{-1/2+\delta})) \ \rightarrow 0$$

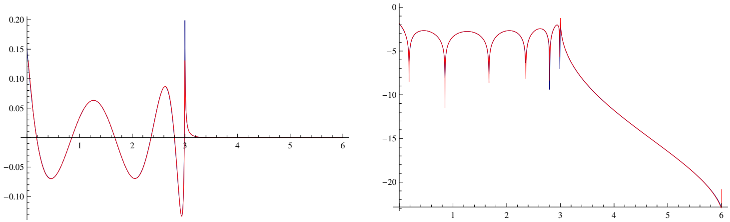
$$\sigma_{-n} = \sqrt{1 - 4e^{-2n\pi K_-/K_+}(1 + O(n^{-1/2+\delta}))} \ \rightarrow 1$$



$a_1 = 0$, $a_2 = 3$, $a_3 = 6$, $a_4 = 12$. Logarithmic plot of the asymptotic (red line) and numerical values (blue dots) of the singular values σ_n tending to zero (left) and $1 - \sigma^2_{-n}$ for the singular values σ_{-n} tending to 1 (right). Numerical value: Mathematica.

Conclusion: the asymptotic expressions are accurate even for small $|n|$.

Results: singular functions



$a_1 = 0$, $a_2 = 3$, $a_3 = 6$, $a_4 = 12$. Plot (left) and logarithmic plot (right) of the singular function g_6. Asymptotic form (red line) and numerical values (blue line). Numerical value: Mathematica.

Conclusion: the asymptotic expressions are accurate even for small $|n|$.

- Context: limited data reconstruction via the differentiated backprojection (DBP).
- Background: asymptotic behaviour of the SVD of the truncated Hilbert transform.
- *Stability estimates for the inversion.*

Inverse problems where σ_n decays exponentially to 0 are severely ill-posed and untractable in practice, they lead typically to

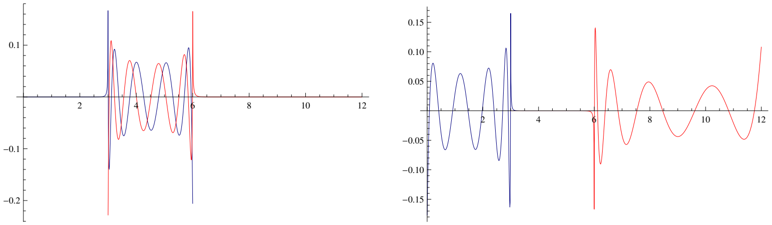
$$\text{reconstruction error } \simeq \frac{C}{|\log \text{noise}|}$$

unless **very strong** prior knowledge is available (e.g. sparsity !).

Examples: backward heat equation, extrapolation of band-limited signals, etc.

Is there then any hope for the truncated Hilbert transform ?
Yes, as shown by the numerical evidence...... but Why ?

Answer: because the singular functions f_n for small σ_n are small within the overlap segment $(a_2, a_3) \Rightarrow$ we expect good stability within the overlap segment, hence stable ROI reconstruction.



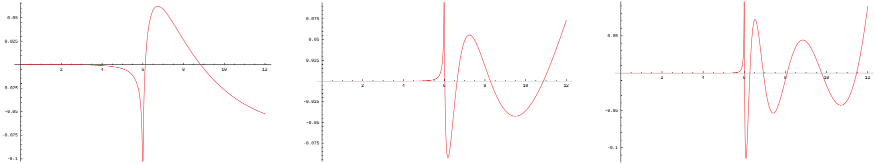
$a_1 = 0$, $a_2 = 3$, $a_3 = 6$, $a_4 = 12$. Blue: g_n, red: f_n. Left: large singular value $\sigma_n \simeq< 1$. Right: small singular value $\sigma_n \simeq> 0$.

How does the stability degrade as $x \to a_3^-$?

Remove a small neighborhood $\mu > 0$ at edge of ROI and study stability on $(a_2, a_3 - \mu)$. We use the asymptotic of f_n to show that

$$||f_n||_{(a_2, a_3-\mu)} = \left(\int_{a_2}^{a_3-\mu} dx |f_n(x)|^2 \right)^{1/2} = \frac{1}{\sqrt{n\pi}} e^{-\beta_\mu n} \left(1 + O(n^{-1/2+\delta})\right)$$

with $\beta_\mu = \frac{\pi}{K_-} \int_{a_3-\mu}^{a_3} \frac{dt}{\sqrt{P(t)}} \simeq Cst\ \sqrt{\mu}$



$\sigma_2 = 1.13 \times 10^{-3}$ $\sigma_4 = 7.71 \times 10^{-7}$ $\sigma_6 = 5.04 \times 10^{-10}$

Singular functions f_n. Note the decreasing amplitude within the overlap ROI $(a_2, a_3) = (3, 6)$ where the Hilbert transform is known.

Stability of the inversion of the truncated Hilbert transform regularized by truncated SVD.

Let

- $f_{ex} \in L^2(a_2, a_4)$ be some object, and we know that $||f_{ex}|| \leq E$ for some E.
- $g_{ex} = H_T f_{ex} \in L^2(a_1, a_3)$ be the noise-free data.
- g be noisy data such that $||g - g_{ex}|| \leq \delta$ for some noise level $\delta > 0$.

Problem: find an estimate of f_{ex} on some interval $a_2 \leq x \leq a_3 - \mu$, with a small $\mu > 0$.

Consider the truncated SVD reconstruction f_M with cut-off index M:

$$f_M = \sum_{n=-\infty}^{M} < g, g_n > \frac{1}{\sigma_n} f_n$$

The reconstruction error is

$$\begin{aligned} f_M - f_{ex} &= \sum_{n=-\infty}^{M} < g - g_{ex}, g_n > \frac{1}{\sigma_n} f_n - \sum_{n=M+1}^{\infty} < g_{ex}, g_n > \frac{1}{\sigma_n} f_n \\ &= \underbrace{\sum_{n=-\infty}^{M} < g - g_{ex}, g_n > \frac{1}{\sigma_n} f_n}_{\text{Statistical error}} - \underbrace{\sum_{n=M+1}^{\infty} < f_{ex}, f_n > f_n}_{\text{Systematic error}} \end{aligned}$$

where we used $< g_{ex}, g_n >=< H_T f_{ex}, g_n >=< f_{ex}, H_T^* g_n >= \sigma_n < f_{ex}, f_n >$.
Using the triangular inequality and all assumptions,

$$||f_M - f_{ex}||_{(a_2, a_3-\mu)} \leq \delta A^{-1} e^{\alpha M} + EC_\mu e^{-\beta_\mu M}$$

with $\alpha = \pi K_+ / K_-$ and $\beta_\mu \simeq Cst \sqrt{\mu}$ the decay rates of σ_n and of $||f_n||_{(a_2, a_3-\mu)}$.

$$||f_M - f_{ex}||_{(a_2,a_3-\mu)} \leq \delta A^{-1} e^{\alpha M} + E C_\mu e^{-\beta_\mu M}$$

Minimize w.r.t. the SVD cut-off $\Rightarrow M(\delta) = \dfrac{1}{\alpha + \beta_\mu} \log\left(\dfrac{EAC_\mu \beta_\mu}{\delta\alpha}\right)$

$\Rightarrow$ Stability estimate

$$||f_{M(\delta)} - f_{ex}||_{(a_2,a_3-\mu)} \leq W_\mu\, E^{\alpha/(\beta_\mu+\alpha)} \underbrace{\delta^{\beta_\mu/(\beta_\mu+\alpha)}}_{\text{Holder continuity}} \xrightarrow{\delta\to 0} 0$$

The inversion is regularized.

Conclusion

$$||f_{M(\delta)} - f_{ex}||_{(a_2,a_3-\mu)} \leq W_\mu\, E^{\alpha/(\beta_\mu+\alpha)} \delta^{\beta_\mu/(\beta_\mu+\alpha)} \xrightarrow{\delta \to 0} 0$$

- Hölder continuity: *error* $\simeq$ *(noise)*$^\eta$ typical of mildly ill-posed problems despite the exponential decay of $\sigma_n \simeq e^{-\alpha n}$.
- But this holds for the error within the segment $(a_2, a_3 - \mu)$ where the Hilbert transform is known, minus some small neighborhood μ.
- The power decreases as $\eta = \beta_\mu/(\beta_\mu + \alpha) \sim \sqrt{\mu}$ for small μ.
- Result obtained for TSVD but it only depends on the used prior constraint $||f_{ex}|| \leq E$. Same dependence for Tikhonov regularization etc.....

Open questions

- Stability bounds for other prior constraint, e.g. $TV(f_{ex}) \leq E$? Preliminary results by Al-Aifari and Steinberger.
- Our bound is pessimistic when the ROI is far from the edge, $a_2 < a_3 - \mu \ll a_3$, since our $\eta \to 1/2$ whereas one expects $\eta \to 1$.

Thank you !