Analytic reconstruction methods for the
Compton camera

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Summary

1. Compton camera: working principle and applications
2. Models for the acquisition process
3. Examples of image reconstruction methods
4. Conclusions
5. Thanks
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Compton camera for SPECT imaging

- **Source of $\gamma$ particles**: emission point $V_0$ and initial energy $E$
- **Scatterer**: first interaction (Compton scattering) at $V_1$ and energy transmitted to an electron $E_1$
- **Absorber**: second interaction at $V_2$ (photoelectric absorption) and energy $E_2$
- **Projection pattern**: integral on the surface of a cone
The data

A $\gamma$ particle emitted at $V_0$ with initial energy $E$

- is Compton scattered at $V_1$ where the energy $E_1$ is transferred to an electron of the scatterer
- then is absorbed by photoelectric effect at the point $V_2$ from the absorber, where the remaining energy $E_2 = E - E_1$ is deposited.

The diffusion angle, also called Compton angle, is then given by

$$\cos \beta = 1 - \frac{me c^2 E_1}{(E - E_1)E}$$
The data

A γ particle emitted at $V_0$ with initial energy $E$

- is Compton scattered at $V_1$ where the energy $E_1$ is transferred to an electron of the scatterer
- then is absorbed by photoelectric effect at the point $V_2$ from the absorber, where the remaining energy $E_2 = E - E_1$ is deposed.

Conversely, the source point $V_0$ lies on the surface of a cone, the Compton cone, with

- apex $V_1$
- axis direction $\vec{\Omega}_2 = \frac{\vec{V}_2 \vec{V}_1}{|\vec{V}_2 \vec{V}_1|}$
- half-opening angle $\beta$
Applications

- Imaging of polyenergetic sources
- Imaging of sources with energies $\sim 1$ MeV

Advantages of the Compton camera:
- devoided of mechanical collimator, its sensitivity is superior to the one of the Anger camera by 1-2 orders of magnitude
- 3D imaging with a single camera
Example of application: hadron therapy

Sphere in PMMA irradiated by a proton beam (140 MeV).

Deposited energy  

Last interaction of $\gamma$ particles that escape the phantom.

Conclusions

- Sources of $\gamma$ particles are imaged,
- with a detector capable to detect particles from arbitrary directions,
- the measured projections being integrals, of the intensity of the source, on conical surfaces.
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**Important parameters**

Influence of the angle of incidence, referred to as $\theta$:

![Diagram of angle of incidence](image1)

Influence of the distance to the detector:

![Diagram of distance to detector](image2)

Let us denote

- $\vec{v}$ the vector of coordinates of $V_1$ in the orthogonal frame $Oxyz$,
- $\vec{u}$ the vector of coordinates of an arbitrary point $M$,
- $\theta$ the angle of incidence on the scatterer, $\cos \theta = \frac{(\vec{u} - \vec{v}) \cdot \vec{n}}{\| \vec{u} - \vec{v} \|}$.

The intensity of the source should be weighted by a function

$$h : (\theta, \| \vec{u} - \vec{v} \|) \mapsto h(\theta, \| \vec{u} - \vec{v} \|) = \frac{\cos \theta}{V_1 M_2^2}.$$
Let $f$ be the intensity of the source. The number of $\gamma$ particles scattered at $V_1$, with an angle $\beta$ and absorbed at $V_2$ is proportional to:

$$C(\vec{u}, \vec{\Omega}_2, \beta) = \int_{\mathbb{R}^3} f(\vec{u}) h(\theta, \|\vec{v} - \vec{u}\|) k(\vec{u}; \vec{v}, \vec{\Omega}_2, \beta) d\vec{u},$$

where $k(\vec{u}; \vec{v}, \vec{\Omega}_2, \beta)$ models the uncertainties on the value of $\beta$ and may include the Klein-Nishina differential cross-section $K(\cos \beta)$. 
Example: no uncertainties and $h(\theta, \| \vec{v} - \vec{u} \|) = 1$

- May be found in [Cree and Bones, 1994], [Basko et al 1998], [Smith 2005].
- The errors on the measures are not accounted for.
- Each point from the scatterer is seen as isolated.

Let us consider spherical coordinates in a local frame with the vertical axis directed by $\vec{\Omega}_2$. For $\phi \in [0, 2\pi)$, let us denote $\vec{\Omega}_1 = \vec{\Omega}_1(\beta, \varphi)$ the generatrices of a Compton cone $C(V_1, V_2, \beta)$. The Compton projections are then:

$$C(\vec{u}, \vec{\Omega}_2, \beta) = K(\cos \beta) \sin \beta \int_0^{2\pi} \int_0^\infty f(\vec{u} + \rho \vec{\Omega}_1) \rho \, d\rho \, d\varphi.$$ 

**Relation to the 3D Radon transform (and redundancy)**

Note that with this model,

$$C(\vec{u}, \vec{\Omega}_2, \pi/2) = K(0) R_3(\vec{\Omega}_2, \vec{v}.\vec{\Omega}_2),$$

where $R_3$ denotes the three-dimensional Radon transform.
Example: no uncertainties and $h(\theta, ||\vec{v} - \vec{u}||) = 1/||\vec{v} - \vec{u}||$

- May be found in [Parra, 2000], [Tomitani and Hirasawa 2002], [Smith 2005].
- The errors on the measures are not accounted for.
- Each point from the scatterer is seen as isolated.

The Compton projections are then:

$$\mathcal{C}(\vec{u}, \Omega_2, \beta) = K(\cos \beta) \sin \beta \int_0^{2\pi} \int_0^\infty f(\vec{v} + \rho \Omega_1)d\rho d\phi,$$

**Relation to cone-beam integrals**

$$p(\vec{v}, \Omega_1) = \int_0^\infty f(\vec{v} + \rho \Omega_1)d\rho$$

are cone-beam integrals of the object. The Compton projections are in this case sum of cone-beam integrals.
Example: no uncertainties and $h(\theta, \| \vec{v} - \vec{u} \|) = \cos \theta$

- May be found in [Maxim et al., 2009].
- The errors on the measures are not accounted for.
- Each point from the scatterer is seen as isolated.

The Compton projections are then:

$$\mathcal{C}(\vec{v}, \vec{\Omega}_2, \beta) = K(\cos \beta) \int (\vec{u} - \vec{v}) \cdot \vec{\Omega}_2 = \| \vec{u} - \vec{v} \| \cos \beta \ f(\vec{u}) \cos \theta d\vec{u}.$$
Example: The uncertainties are modelled and
\[ h(\theta, \| \vec{v} - \vec{u} \|) = \frac{1}{\| \vec{v} - \vec{u} \|^2} \]

- May be found in [Hirasawa and Tomitani, 2003].
- Gaussian model for the uncertainties, \( \beta \in [\beta_1, \beta_2] \),

\[
k(\vec{u}; \vec{v}, \overrightarrow{\Omega}_2, \beta) = K(\cos \beta) \frac{1}{\sqrt{2\pi}} \exp \left( - \frac{(\cos(\vec{u} - \vec{v}, \overrightarrow{\Omega}_2) - \cos \beta)^2}{2(\sigma \sin \beta)^2} \right).
\]

The Compton projections are then:

\[
C(\vec{v}, \overrightarrow{\Omega}_2, \beta) = \int_{\beta_1}^{\beta_2} \left( \int_0^{2\pi} p(\vec{v}, \overrightarrow{\Omega}_1) d\varphi \right) k(\vec{v} + \overrightarrow{\Omega}_2(\tilde{\beta}, 0); \vec{v}, \overrightarrow{\Omega}_2, \beta) \sin \tilde{\beta} d\tilde{\beta},
\]

**Relation to cone-beam integrals**

The Compton projections are again sum of cone-beam integrals.
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State of the art

- Methods where the 3D Radon projections are calculated: [Basko et al, 1998], [Smith, 2005]
- Methods where the cone-beam projections are calculated: [Parra, 2000], [Tomitani and Hirasawa, 2002], [Hirasawa and Tomitani, 2003]
- Direct inversion: [Cree and Bones, 1994], [Maxim et al, 2009], [Lojacono et al, 2011], [Maxim, 201?]
- Methods using series expansions in spherical harmonics and Lagrange polynomials: [Basko et al, 1998], [Parra, 2000], [Tomitani and Hirasawa, 2002], [Hirasawa and Tomitani, 2003]
- Methods using the Hilbert transform: [Smith, 2005]
- Methods that may be compared to the Fourier-slice theorem and FBP: [Cree and Bones, 1994], [Maxim et al, 2009], [Lojacono et al, 2011], [Maxim, 201?].
[Tomitani and Hirasawa, 2002]

- The Compton projections

\[ C(\vec{v}, \Omega_2, \beta) = K(\cos \beta) \sin \beta \int_0^{2\pi} \int_0^\infty f(\vec{v} + \rho \Omega_1) d\rho d\varphi, \]

are sums of cone-beam integrals

\[ p(\vec{v}, \Omega_1) = \int_0^\infty f(\vec{v} + \rho \Omega_1) d\rho. \]

- The aim of the method is to calculate, independently for each given \( \vec{v} \), the values \( p(\vec{v}, \Omega_1) \) for \( \Omega_1 \in S \).

- Equivalent to filtering after back-projection on conical surfaces.
[Tomitani and Hirasawa, 2002] : reconstruction of cone-beam projections

The reconstruction formula is:

\[ p(\vec{v}, \Omega_1) = \int_{\beta_1}^{\beta_2} \int_S k^{-1}(\Omega_2, \Omega_1), \cos \tilde{\beta}) \mathcal{G}(\vec{v}, \Omega_2, \tilde{\beta}) d\Omega_2 d(\cos \tilde{\beta}), \]

with

\[ k^{-1}(s, t) = \frac{1}{4\pi} \sum_{n=0}^{\infty} \frac{2n+1}{H_n} P_n(s)P_n(t), \quad s, t \in [-1, 1] \]

where \( P_n \) are Lagrange polynomials and

\[ H_n = \int_{\beta_1}^{\beta_2} K(\cos \beta)P_n^2(\cos \beta)d(\cos \beta). \]
Tomitani and Hirasawa, 2002: Reconstruction of one slice from a spherical source.
With
\[ P(\vec{\Omega}_2, \vec{v} \cdot \vec{\Omega}_2) = -\text{p.v.} \int_0^\pi C(\vec{v}, \vec{\Omega}_2, \beta) \frac{1}{\cos \beta} d\beta, \]

one has:
\[ P(\vec{\Omega}_2, \ell) = \text{p.v.} \int_{-\infty}^\infty R_3(\vec{\Omega}_2, t) \frac{1}{\ell - t} dt, \]

which is the Hilbert transform of the 3D Radon transform \( R_3(\vec{\Omega}_2, \cdot) \).
[Smith, 2005] : some results

Detector in $4\pi$, deterministic projections:

(a) original image

(b) reconstructed image

slice at $z=0$  
slice at $y=0$

(MSc thesis of Hussein Banjak, co-supervised by Rolf Clackdoyle)
\[ C(\vec{v}, \vec{\Omega}_2, \beta) = K(\cos \beta) \int (\vec{u} - \vec{v}) \cdot \vec{\Omega}_2 = \parallel \vec{u} - \vec{v} \parallel \cos \beta \] 

\[ f(\vec{u}) \cos \theta d\vec{u}, \]

\[ \vec{\Omega}_2(0, \pi/4), \quad \beta = \pi/3 \]

\[ V_1 \text{ in the plane } z = 0 \]

\[ \vec{\Omega}_2(\pi/5, \pi/4), \quad \beta = \arcsin(\sqrt{\frac{3}{2}} \cos \frac{\pi}{5}) \]

\[ V_1 \text{ in the plane } z = 0 \]
We show that
\[
f(x, y, z) = 2\pi \int_0^{\pi} \int_{-\infty}^{\infty} \left( \int_0^{\infty} \mathcal{P}_{\tau, \delta} f(\rho) J_0(2\pi z \tau \rho) d\tau \right) e^{2i\pi \rho(-x \sin \delta + y \cos \delta)} |\rho|^3 d\rho d\delta,
\]
with
\[
\mathcal{P}_{\tau, \delta} = \frac{1}{K(\cos \beta)} \mathcal{R}_\delta + \frac{\pi}{2} \mathcal{C}(\cdot, \Omega(\alpha, \delta), \beta)
\]
for \( \tau = \frac{\sin \beta}{\sqrt{\cos^2 \alpha - \sin^2 \beta}} \),
and \( J_0 \) the Bessel function.
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Redundancy in the data set

- The image space is three-dimensional.
- The data space is six-dimensional: 3 dof for the apex, 2 dof for the axis, 1 dof for the Compton angle.
- However, for a common acquisition geometry, not all the data are acquired, leading to artifacts in the reconstructed images.
Conclusions

- Several classes of methods were proposed in the literature.
- The Compton data are redundant but ...
- Except for maybe impractical acquisition geometries, all the methods suffer from the absence of a part of the data sets.

Open questions:

- Is there any possibility for a local method, knowing that Compton projections are surface projections like the 3D Radon transform?
- There are several models for the direct problem. Which one would be the best?
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