ROI Reconstruction in CT

From Tomo reconstruction in the 21st centery, IEEE Sig. Proc. Magazine (R.Clackdoyle M.Defrise)

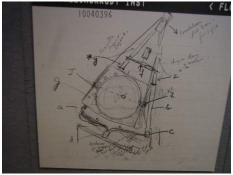
L. Desbat

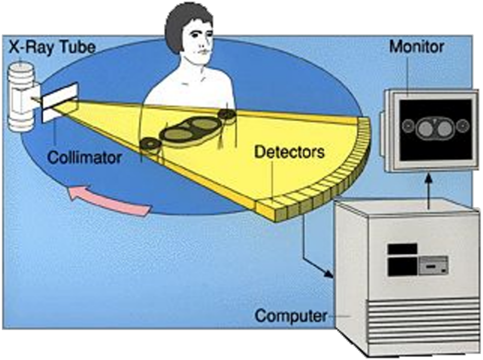
TIMC-IMAG

September 10, 2013

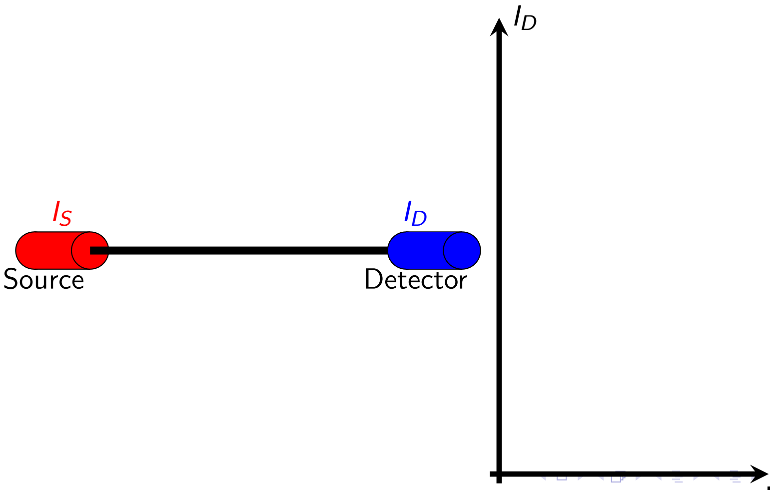
Outline

CT scanner: principle

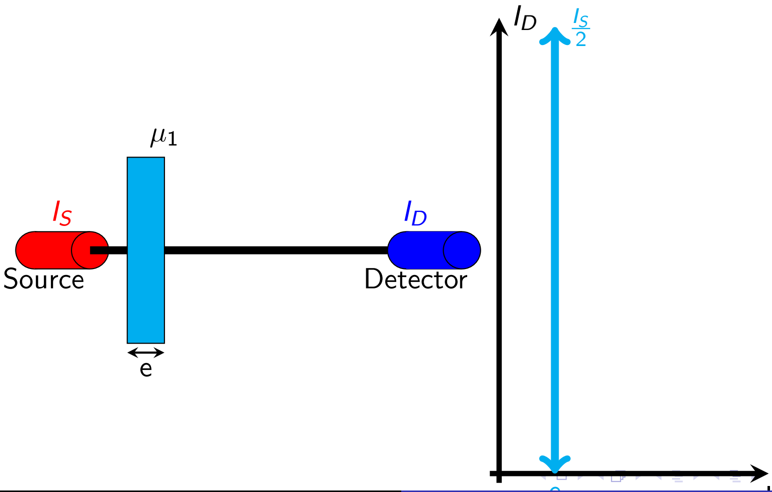




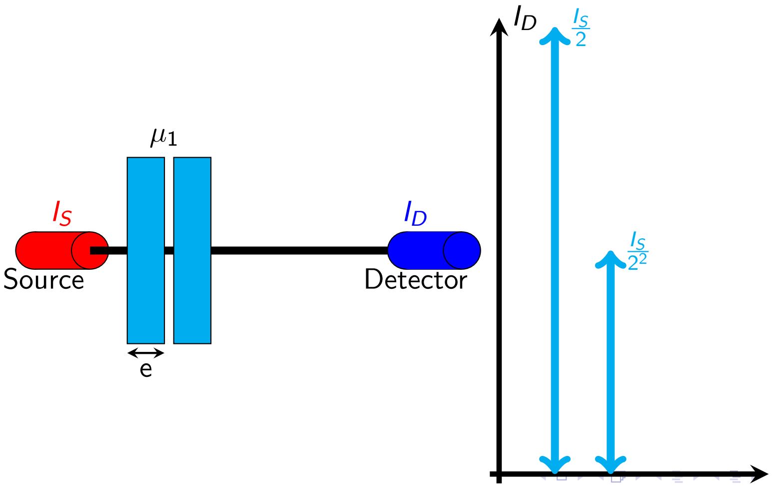
X-ray attenuation



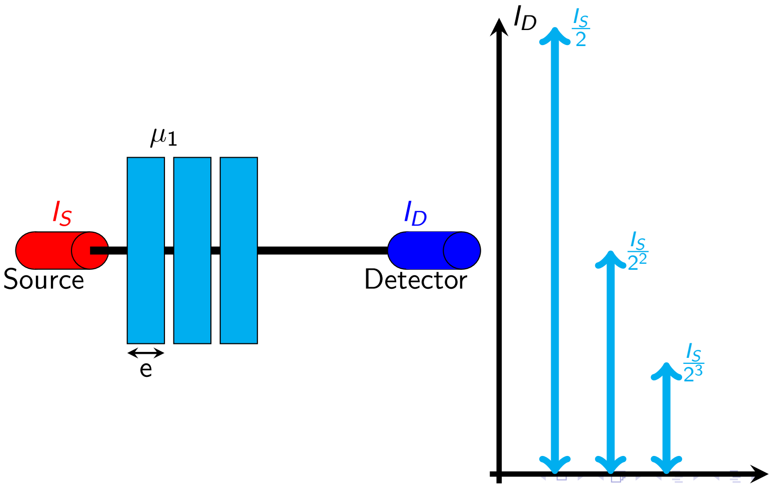
X-ray attenuation



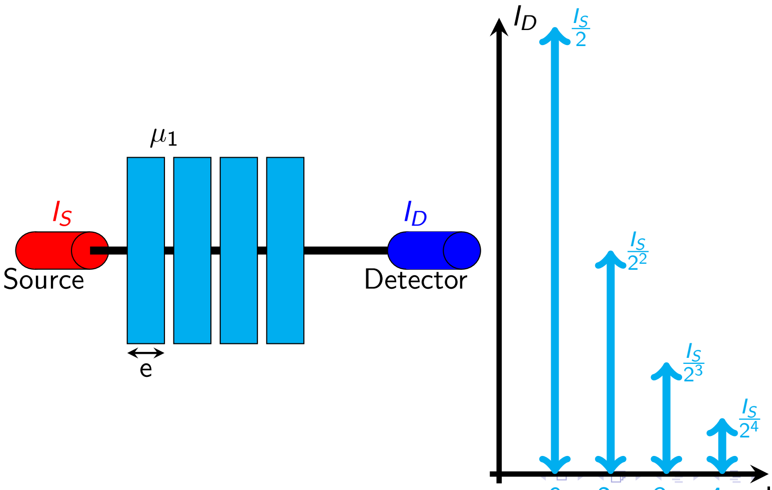
X-ray attenuation



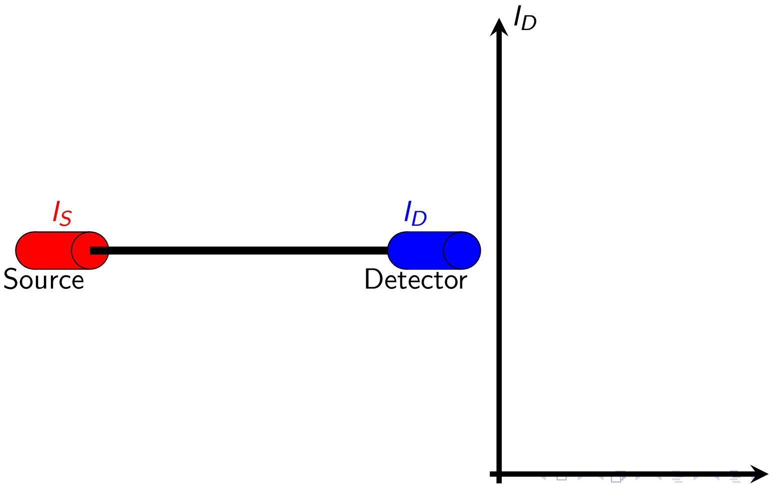
X-ray attenuation



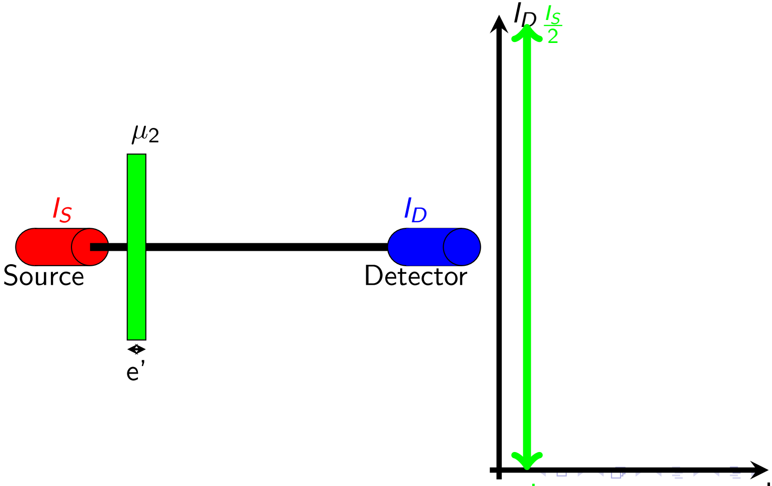
X-ray attenuation



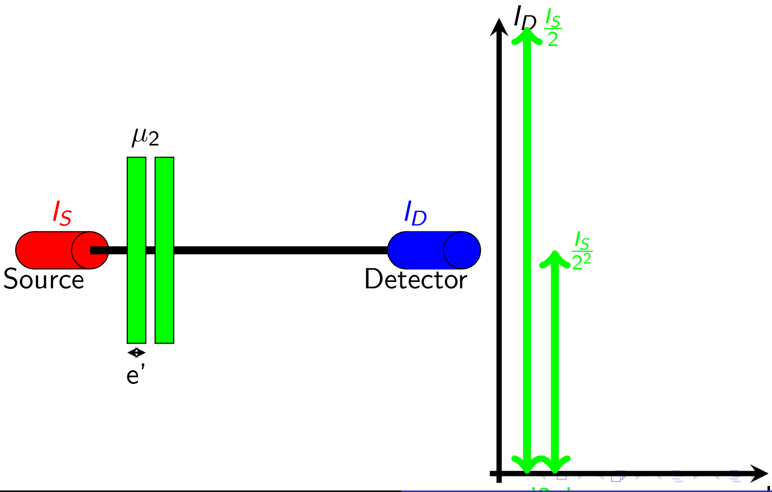
X-ray attenuation



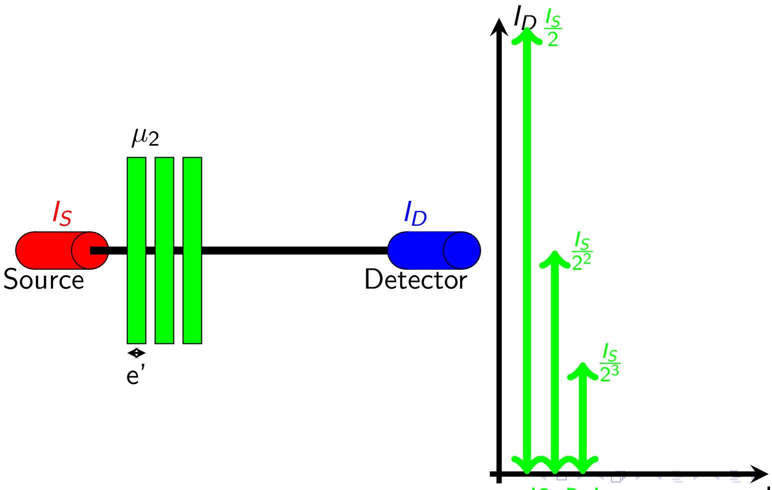
X-ray attenuation



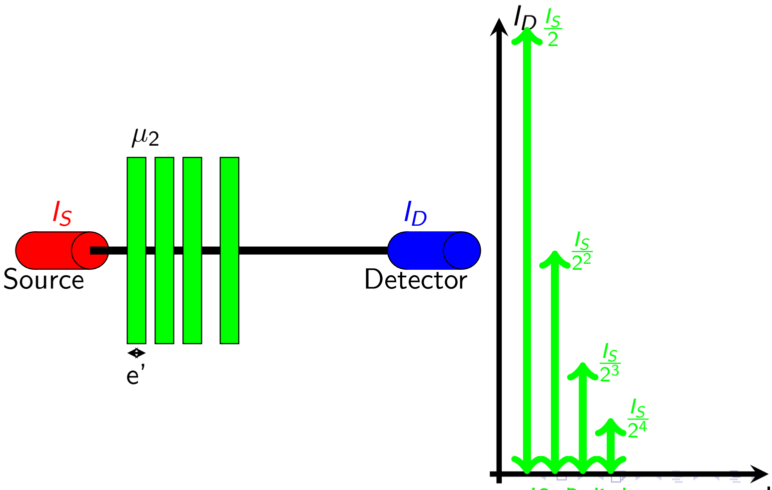
X-ray attenuation



X-ray attenuation

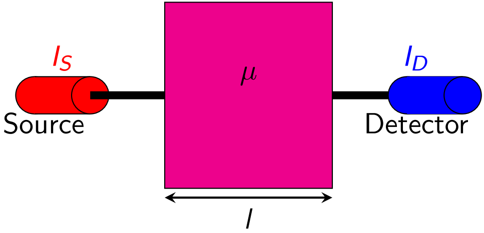


X-ray attenuation



X-ray attenuation

X-ray attenuation



Lambert-Beer's Law

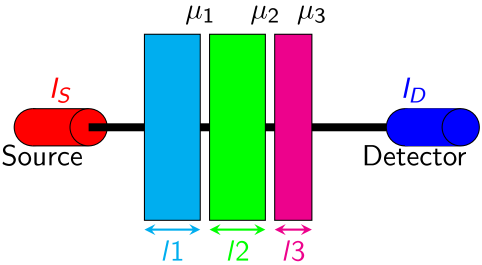
$$I_D(x) = I_S e^{-\mu l}$$

I_S : intensité initiale

μ : coefficient d'atténuation

l : épaisseur du tissus

X-ray attenuation



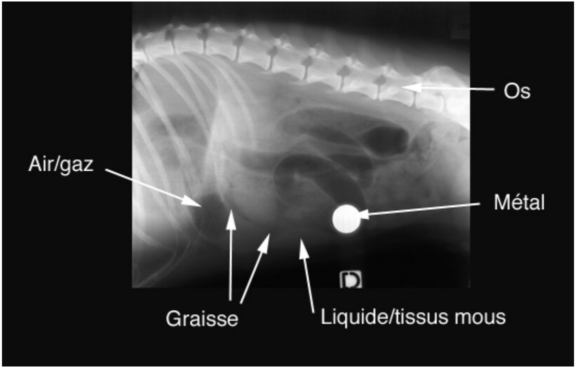
Lambert-Beer's Law

$$I_D = I_S e^{-(\mu_1 l_1 + \mu_2 l_2 + \mu_3 l_3)}$$

$$I_D = I_S e^{-\sum_i \mu_i dl_i}$$

$$I_D = I_S e^{-\int_S^D \mu(l) dl}$$

x-ray image formation



From Radiology to Tomography

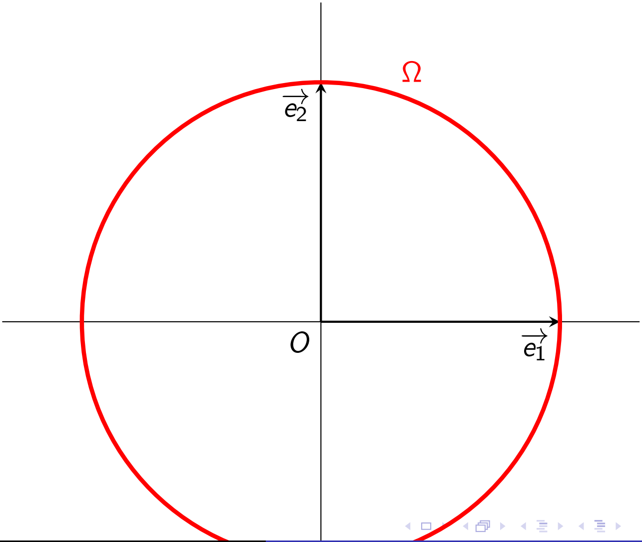
- Radon Transform

$$I_D = I_S e^{-\int_S^D \mu(l) dl}$$

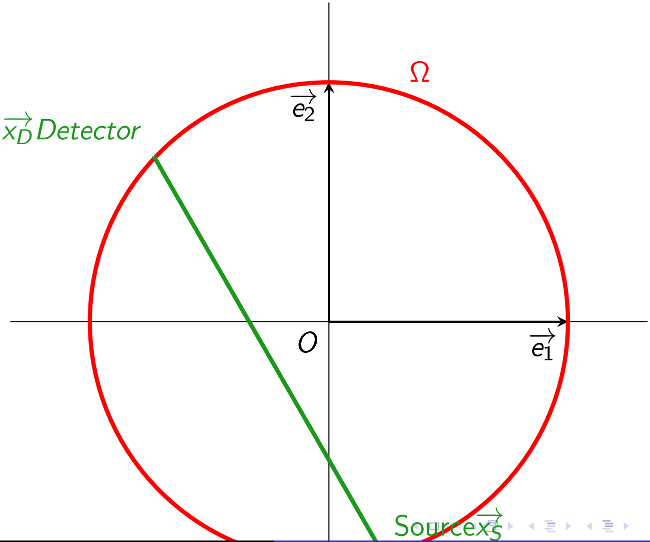
$$-ln\left(\frac{I_D}{I_S}\right) = \int_S^D \mu(l) dl$$

- With a CT we measure the integral of μ over lines

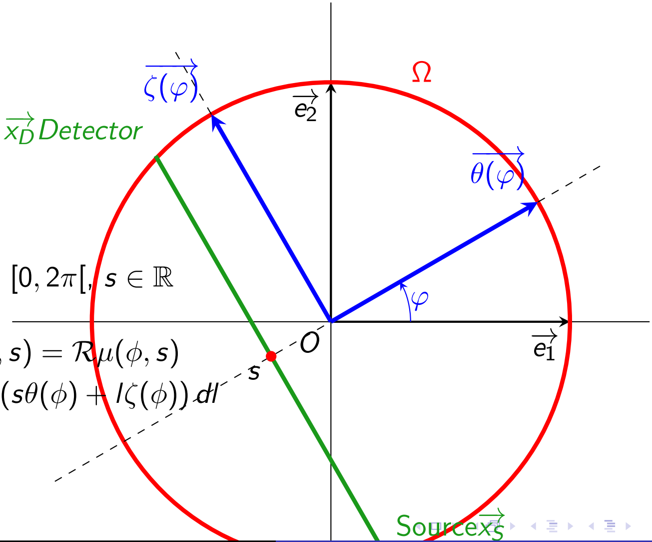
Radon Transform parametrization



Radon Transform parametrization



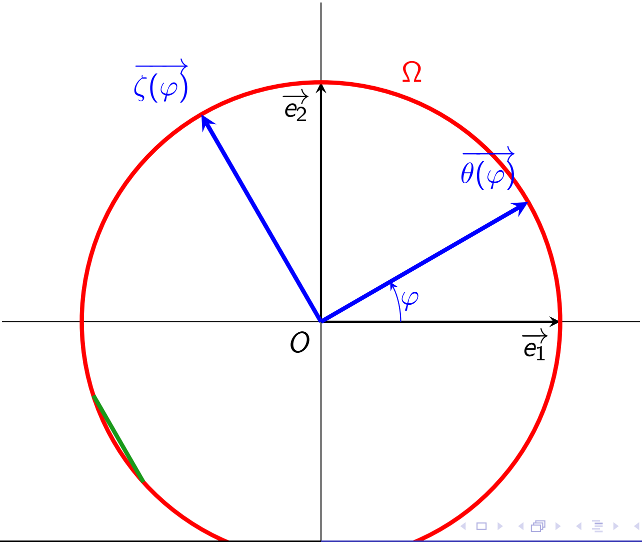
Radon Transform parametrization



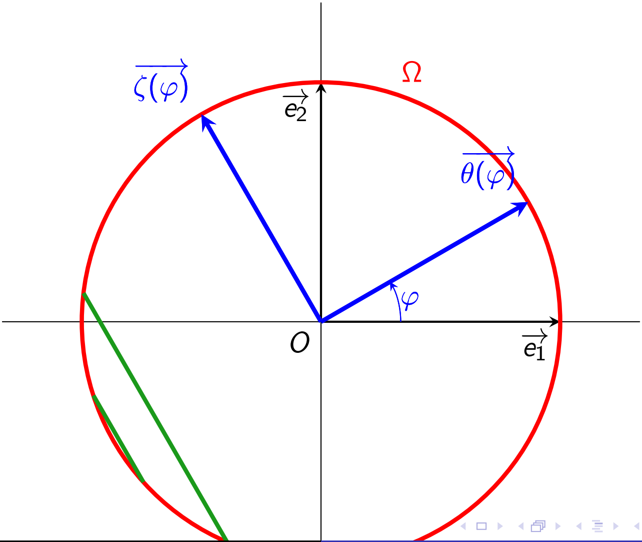
$$\phi \in [0, 2\pi[, s \in \mathbb{R}$$

$$p(\phi, s) = \mathcal{R}\mu(\phi, s) = \int_{\mathbb{R}} \mu\left(s\theta(\phi) + l\zeta(\phi)\right) dl$$

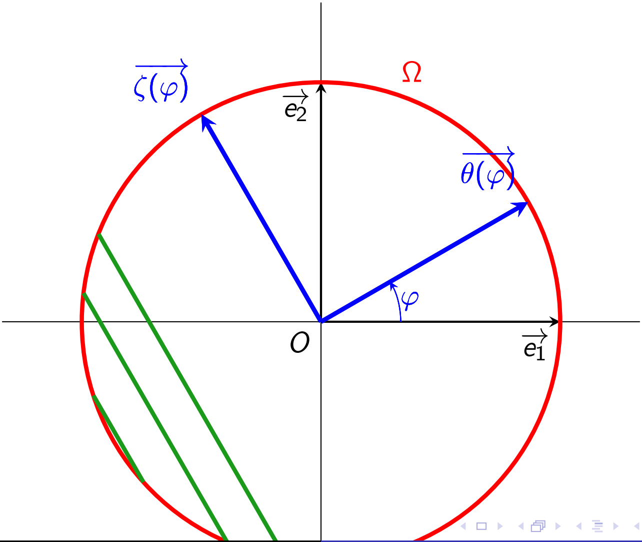
Radon Transform



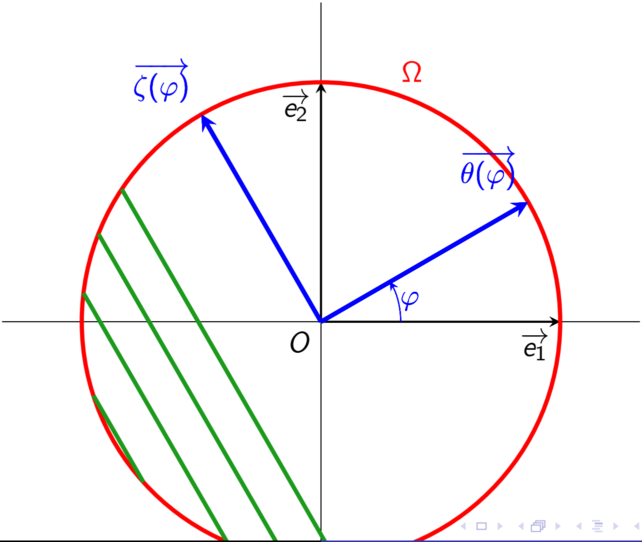
Radon Transform



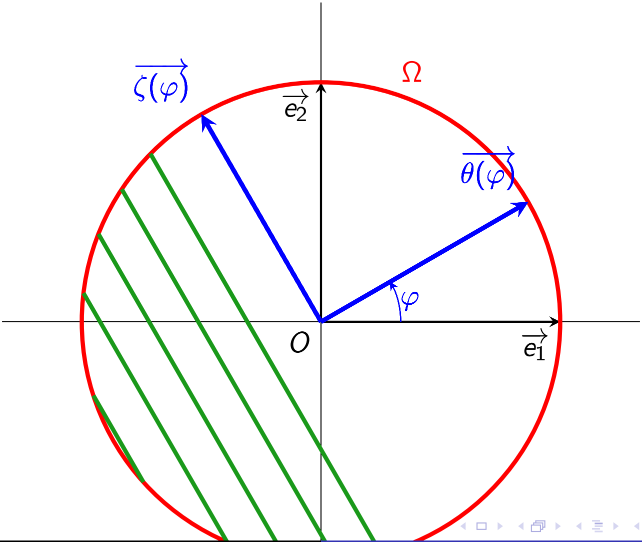
Radon Transform



Radon Transform



Radon Transform



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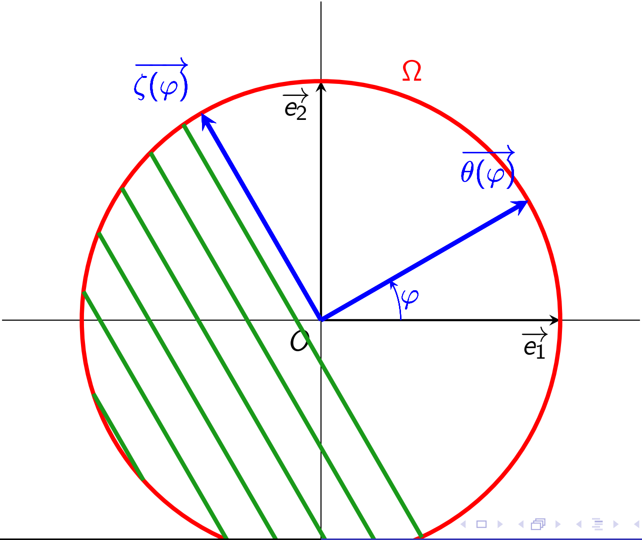

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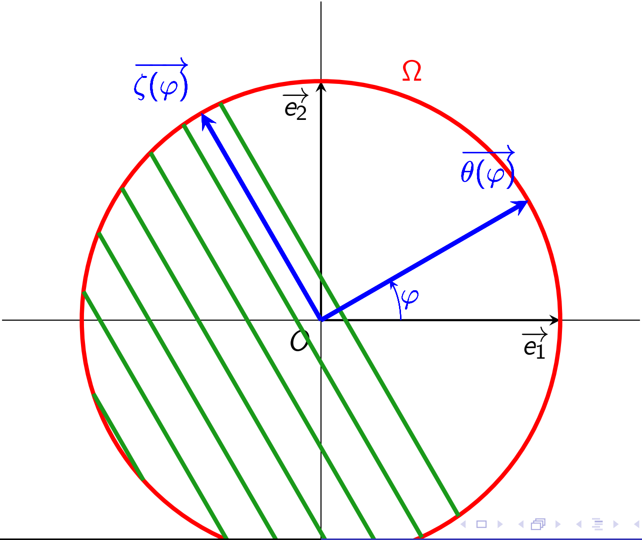
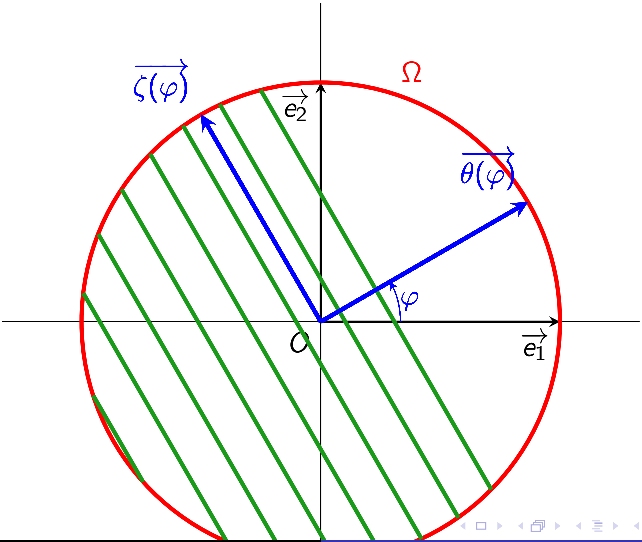
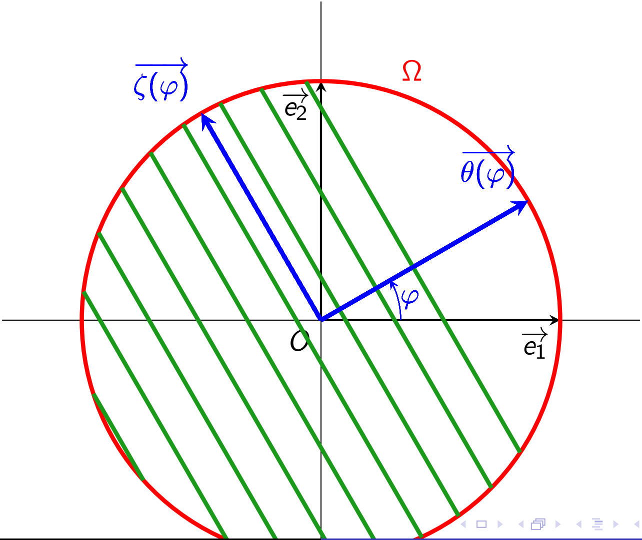


Radon Transform



Radon Transform



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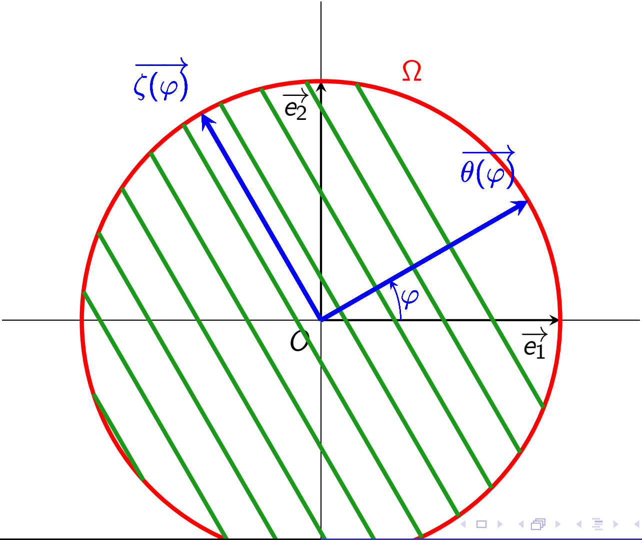

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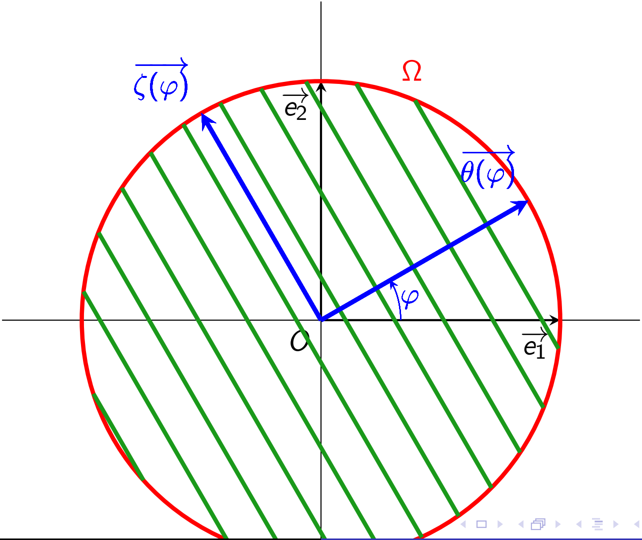

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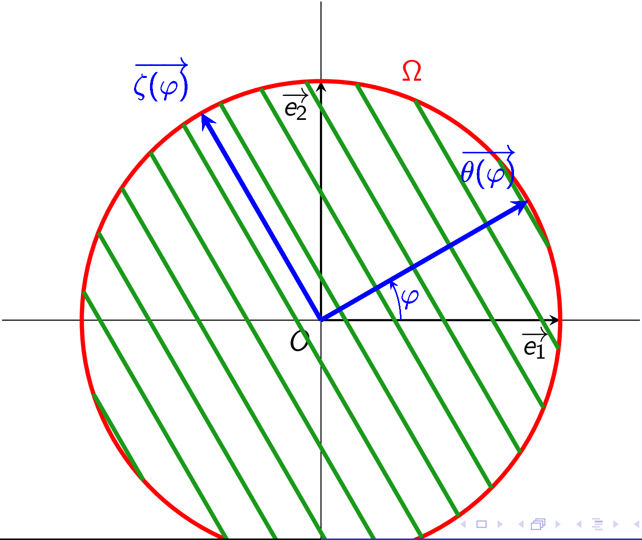
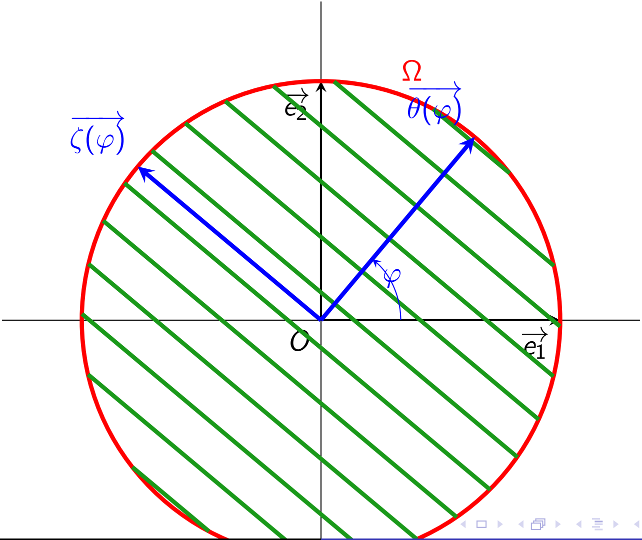


Radon Transform



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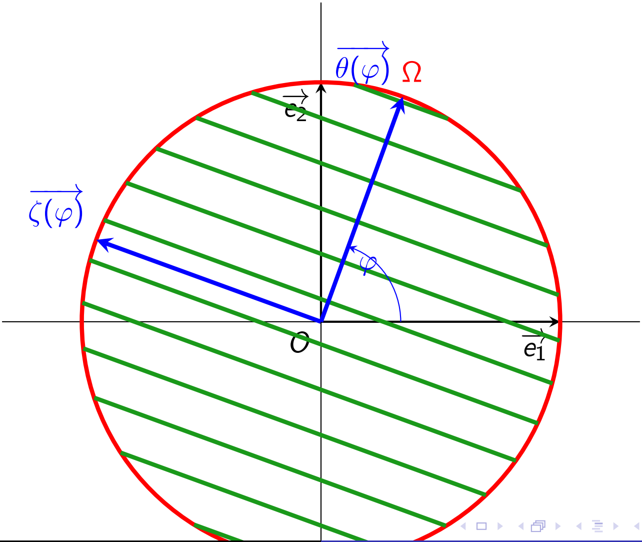
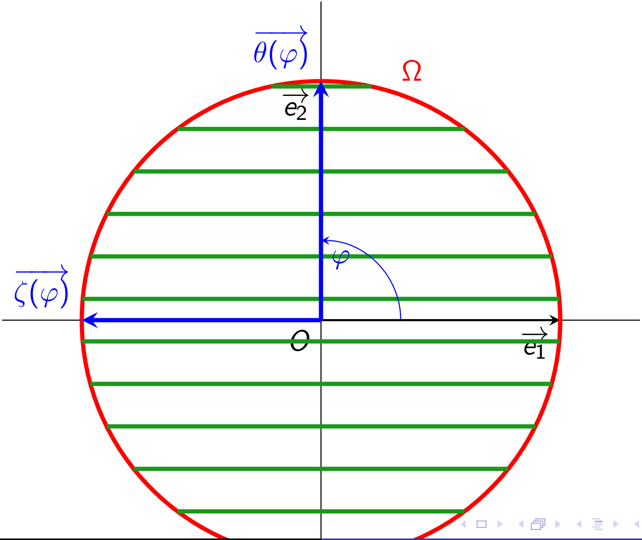


Radon Transform



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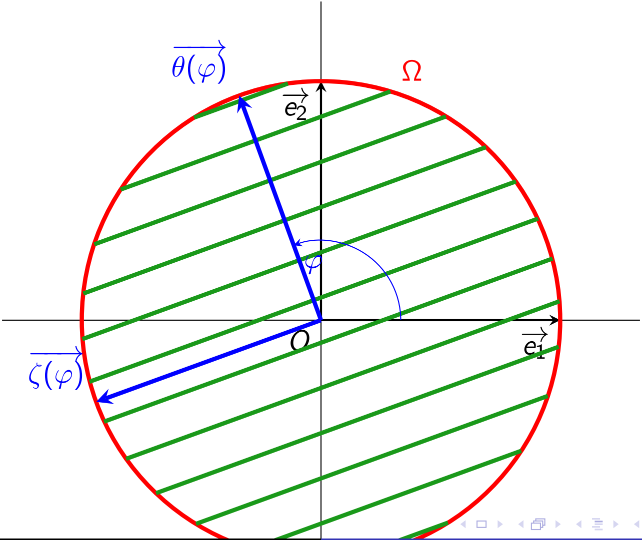
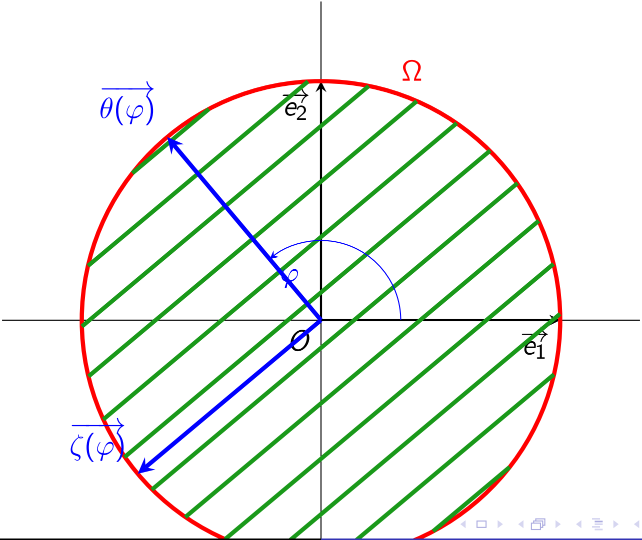
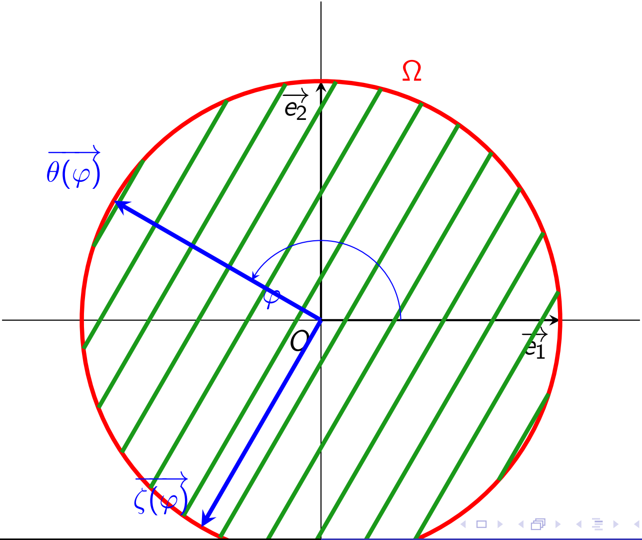


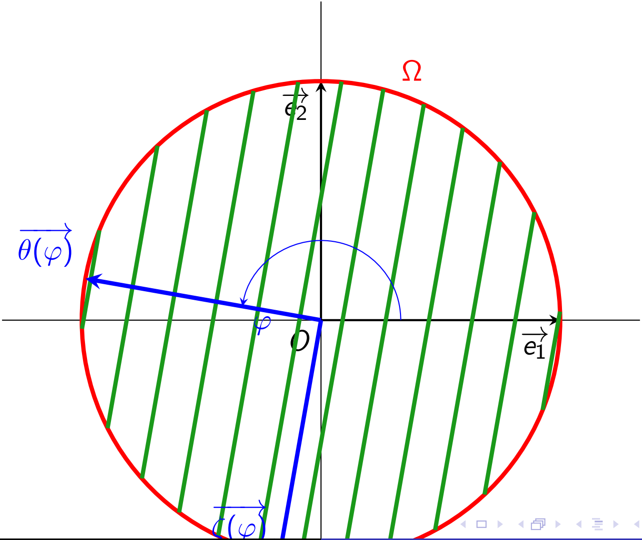
Radon Transform



Radon Transform



Radon Transform



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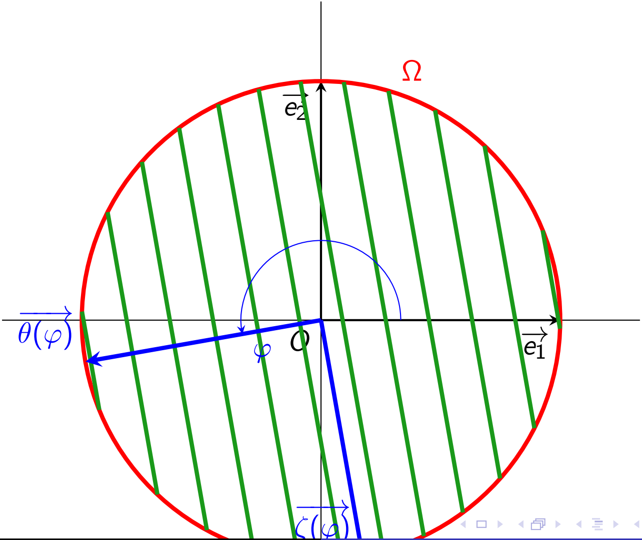
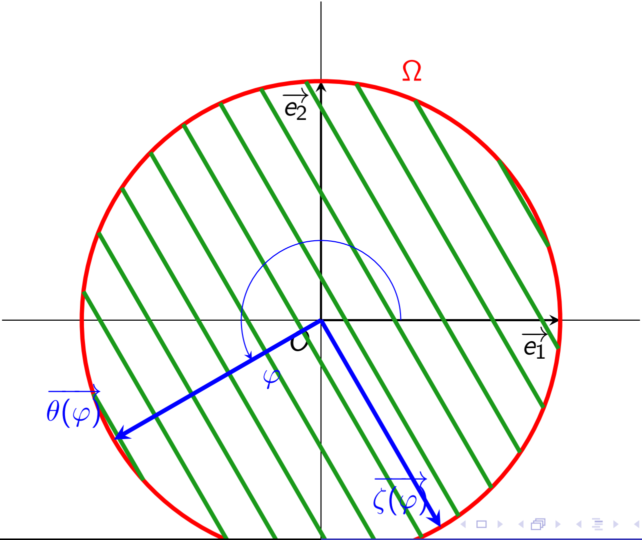


Radon Transform



Center Slice Theorem

Definition

$$\mathcal{R}_\theta \mu f\,(s) \stackrel{\text{def}}{=} \mathcal{R}\mu f\,(\theta, s) \tag{1}$$

Theorem

Let $\mu \in \mathbb{L}^1(\mathbb{R}^2)$ *then*

$$\widehat{\mathcal{R}_\theta \mu}(\sigma) = \hat{\mu}(\sigma\theta)$$

Proof of Center Slice Theorem

Proof.

$$\begin{aligned}
\widehat{\mathcal{R}_\theta \mu}(\sigma) &= \int_{\mathbb{R}} \mathcal{R}_\theta \mu(s) e^{-2i\pi\sigma s} ds \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \mu\left(s\theta + l\zeta\right) dl e^{-2i\pi\sigma s} ds \\
&= \int_{\mathbb{R}^2} \mu(x) e^{-2i\pi\sigma\theta\cdot x} dx \\
&= \hat{\mu}(\sigma\theta)
\end{aligned}$$

where we made the change of variables (s, l) to (x_1, x_2) i.e. $x = s\theta + l\zeta$. This is a rotation thus $dlds = dx_1 dx_2$ □

Filtered Back Projection

Theorem

Let $\mu \in \mathbb{L}^1(\mathbb{R}^2)$ *sufficiently smooth then*

$$\mu(x) = \int_0^{\pi} \int_{\mathbb{R}} \widehat{\mathcal{R}_\theta \mu}(\sigma)|\sigma| e^{2i\pi\sigma x \cdot \theta} d\sigma d\phi$$

Filtered Back Projection, proof

Proof.

$$\begin{aligned}\mu(x) &= \int_{\mathbb{R}^2} \hat{\mu}(\xi) e^{2i\pi\xi\cdot x} d\xi \\ &= \int_0^\pi \int_{\mathbb{R}} \hat{\mu}(\sigma\theta) e^{2i\pi\sigma\theta\cdot x} |\sigma| d\sigma d\phi \\ &= \int_0^\pi \int_{\mathbb{R}} \widehat{\mathcal{R}_\theta \mu}(\sigma) |\sigma| e^{2i\pi\sigma\theta\cdot x} d\sigma d\phi\end{aligned}$$

where we have made the polar change of variable $\xi = \sigma\theta(\phi)$, $(\phi, \sigma) \in [0\pi[\times\mathbb{R}$, thus $d\xi_1 d\xi_2 = |\sigma| d\sigma d\phi$, and we have used theorem 2. □

Non Local Filter

We define p_H the Hilbert transform of the parallel projection p

$$p_H(\phi, s) = \int_{-\infty}^{+\infty} p(\phi, t) h(s - t) dt \text{ where } h(u) = \frac{1}{\pi u}$$

and $\hat{h}(\sigma) = -i\mathrm{sgn}(\sigma)$ (distribution). The ramp filtering p_R of p is

$$p_R(\phi, s) = \frac{1}{2\pi} \frac{\partial}{\partial s} p_H(\phi, s)$$

The filter $p_\phi(s) \stackrel{\mathcal{F}}{\longrightarrow} \widehat{p_\phi}(\sigma) \stackrel{\text{filter}}{\longrightarrow} \widehat{p_\phi}(\sigma)|\sigma| \stackrel{\mathcal{F}^{-1}}{\longrightarrow} p_{\phi_{\text{filtered}}}(s)$ is non local !!! $|\sigma| = \frac{1}{2\pi}(2i\pi\sigma)(-i\mathrm{sgn}(\sigma))$ is the Hilbert filtering composed by the derivation.

Fan Beam geometry

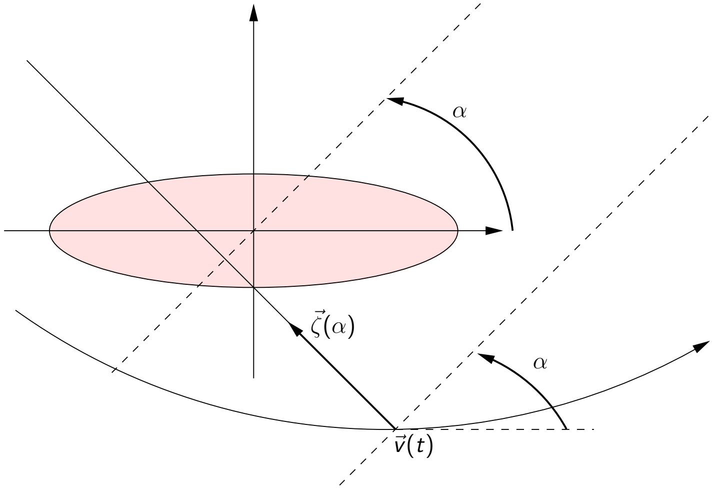


Figure: The Fan Beam variables (t, α)

Fan Beam geometry

We first define the source trajectory along a curve

$$\begin{array}{rcl} v \ : \ \mathcal{C} & \longrightarrow & \mathbb{R}^2 \\ t & \longrightarrow & v(t) \end{array}$$

The fan-beam data are then defined by

$$g\left(v_t, \alpha\right) = \int_0^{+\infty} \mu\left(v_t + l\zeta(\alpha)\right) dl \qquad (2)$$

We remark that

$$p(\phi, s) = g(v_t, \phi) + g\left(v_t, \phi + \pi\right) \text{ where } s = v_t \cdot \theta(\phi)$$

Fan Beam Inversion

We consider the circular trajectory, $v_t = (-R_v \cos t, -R_v \sin t)$,

Theorem

Let $\mu \in \mathbb{L}^1(\mathbb{R}^2)$ sufficiently smooth then

$$\mu(x) = \frac{1}{2} \int_0^{2\pi} \frac{1}{||x - v_t||^2} g_{WF}\left(v_t, \arg(x - v_t)\right) dt$$

where

$$g_{WF}\left(v_t, \phi\right) = \int_{t-\pi/2}^{t+\pi/2} R_v \cos(\psi - t) g(v_t, \psi) r\left(\sin(\phi - \psi)\right) d\psi$$

where r is the ramp filter ($\hat{r}(\sigma) = |\sigma|$).

Proof ... change of variables

Short Scan

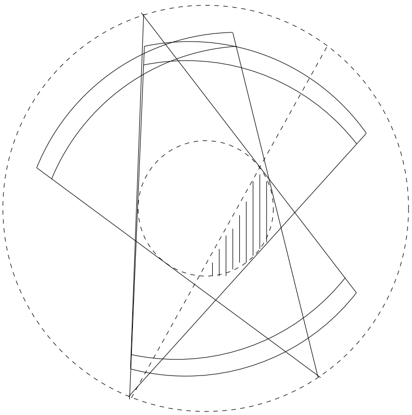


Figure: Short scan with (Parker) weight is possible....

Trajectories, small detectors

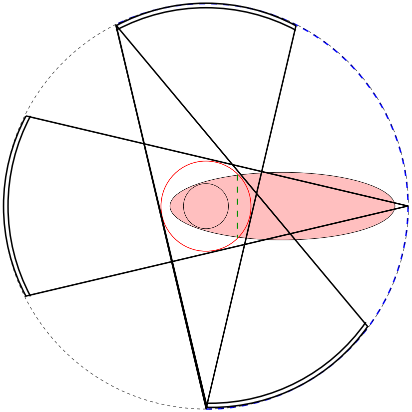


Figure: Small detector yields truncated data.

Hilbert and Fan Beam

We define g_H the Hilbert transform of the fan beam projection g

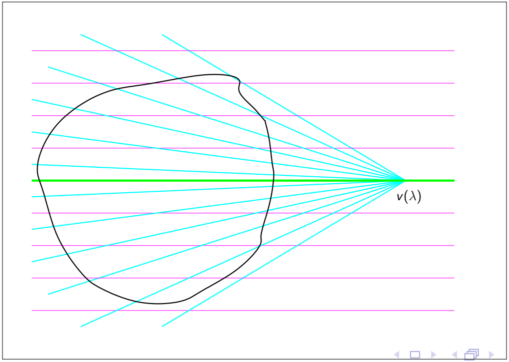
$$g_H(v_t, \phi) = \int_0^{2\pi} g_H(v_t, \psi) h\left(\sin(\phi - \psi)\right) d\psi \text{ where } h(u) = \frac{1}{\pi u}$$

The parallel Fan Beam Hilbert Projection Equality

Theorem

$$p_H(\phi, v_t \cdot \theta) = g_H(v_t, \phi) \qquad (3)$$

(idea compute $p_H(\phi, s)$ from $g_H(v_t, \phi)$ with $v_t \cdot \theta = s$)



New Reconstruction Conditions

Recall

$$p_R(\phi, s) = \frac{1}{2\pi}\frac{\partial}{\partial s} p_H(\phi, s)$$

with

$$p_H(\phi, s) = g_H(v_t, \phi)$$

Theorem

The point x can be reconstructed from FB non truncated projections provided a fan beam vertex can be found on each line passing through x.

New Reconstruction Formula

$$\mu(x) = \frac{1}{2}\int_{\mathcal{C}} \frac{1}{||x - v_t||} w\left(v_t, \arg(x - v_t)\right) g_F\left(v_t, \arg(x - v_t)\right) dt$$

where

$$g_F\left(v_t, \phi\right) = \frac{1}{2\pi}\int_{t-\pi/2}^{t+\pi/2} h\left(\sin(\psi - \phi)\right) \frac{\partial g}{\partial t}(v_t, \psi) d\psi$$

where h is the hilbert filter.

Very Short Scan

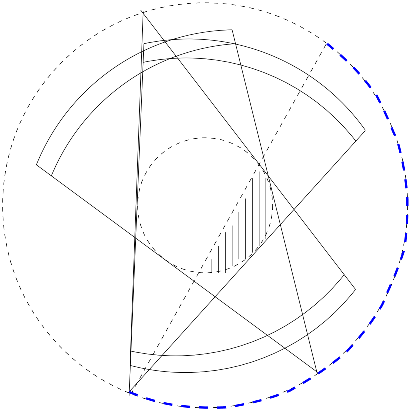


Figure: Very Short Scan....

Virtual Fean Beam

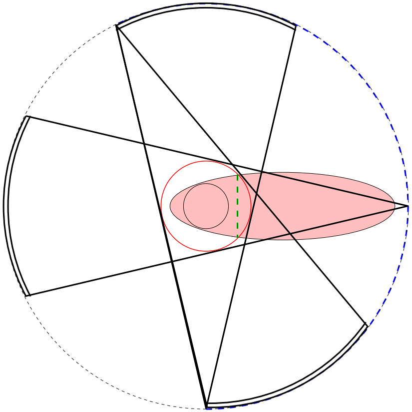


Figure: Data truncation: spine....

Idea

The idea of the Differentiated Backprojection is to compute the Hilbert transform of μ along a direction α from the back projection of the devivation of the projection. μ is then reconstructed from the inversion of the Hilbert transform.

Truncated projections

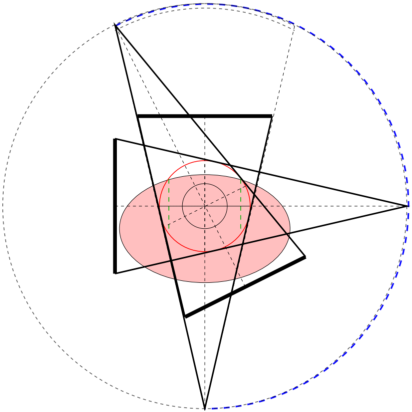


Figure: Data truncation: spine....