# Asymptotic Analysis of the Single-Pinhole Transform in Fourier Space for Efficient Acquisition

Alvin Ihsani[1] Troy Farncombe[2]

[1]Computational Engineering and Science
McMaster University

[2]Radiology
McMaster University

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# Motivation

The recent introduction of organ specific SPECT imaging systems has focused attention on using arrays of small cameras in combination with pinhole collimators rather than conventional parallel hole collimators. While offering advantages in terms of imaging speed and detection sensitivity, cost issues inevitably arise with increased numbers of detector arrays and compromises must be made between image quality (i.e. counts) and the number of detectors (i.e. cost).

# Project Goal

**Objective**

Construct a single-pinhole SPECT imaging device that utilizes the smallest number of detector heads while retaining the desired spatial resolution.

- One approach to achieve this goal is to mathematically formulate a single-pinhole transform (SPT) and analyze its properties in Fourier space so as to obtain a domain of frequencies which allow for the reconstruction of a function without introducing artifacts. To accomplish this two aspects will be treated:
    - sampling lattices which will enable the exact reconstruction of a function;
    - asymptotic properties of the function/transform to determine the optimal sampling lattice.

# Sampling

Under what conditions can a function be recovered from its values on a discrete lattice?

Let a lattice in $\mathbb{R}^n$ be described by a nonsingular matrix $W = (w_1, \ldots, w_n) \in \mathbb{R}^{n \times n}$ as

$$L_W = \left\{ x \in \mathbb{R}^n \middle| x = \sum_{i=1}^{n} k_i w_i, k_i \in \mathbb{Z} \right\} = W\mathbb{Z}^n$$

with a reciprocal lattice $L_W^{\perp} = L_{2\pi W^{-\top}}$.

# Sampling of Bandlimited Functions

## Theorem

*Let* $f \in L_2(\mathbb{R}^n)$*, and let* $\hat{f} = 0$ *outside some compact set* $K \subseteq \mathbb{R}^n$*. Assume that the translates* $\overset{\circ}{K} + \xi \cap \overset{\circ}{K} + \xi' = \emptyset$ *for* $\xi, \xi' \in L_W^{\perp}$ *and* $\xi \neq \xi'$*, then* $f$ *is uniquely determined by its values on* $L_W$ *and*

$$\int_{\mathbb{R}^n} f(x)dx = |\det(W)| \sum_{x \in L_W} f(x)$$

*and*

$$||f||^2_{L_2(\mathbb{R}^n)} = |\det(W)| \sum_{x \in L_W} |f|^2(x).$$

- The trapezoidal rule for integration is exact.
- The reconstruction from the samples is stable in an $L_2$ sense.

# Sampling Periodic Functions

Let $f$ be in $\mathbb{R}^n$ with $n$ linear independent periods $p_1, \ldots, p_n \in \mathbb{R}^n$ and let $P = (p_1, \ldots, p_n)$ with lattices $L_P$ and $L_P^\perp$, then $f$ can be viewed as a function in a quotient group $\mathbb{R}^n / L_P$ described by

$$P[0,1)^n = \left\{ x \in \mathbb{R}^n \middle| x = \sum_{i=1}^{n} \lambda_i p_i, \quad \lambda_i \in [0,1) \right\}.$$

The Fourier transform is computed as

$$\hat{f}(\xi) = \frac{1}{|\det(P)|} \int_{\mathbb{R}^n / L_P} e^{-i\langle x, \xi \rangle} f(x) dx, \quad \xi \in L_P^\perp.$$

Since the Fourier transform is defined on a discrete set of points then the inverse Fourier transform is

$$\tilde{f}(x) = \sum_{\xi \in L_P^\perp} e^{i\langle x, \xi \rangle} \hat{f}(\xi).$$

# Sampling Periodic Functions

Provided that the function $f$ is periodic we need a sampling lattice $L_W$ such that $L_P \subseteq L_W$ (equivalently $L_W^\perp \subseteq L_P^\perp$) that has the same periods.

$\Rightarrow$ The function $f$ must be sampled at a rate that its periods dictate.

## Theorem

*Let $f \in L_2(\mathbb{R}^n/L_P)$ with $\hat{f} = 0$ outside a finite set $K \subseteq L_P^\perp$. Assuming $K + \xi \cap K + \xi' = \emptyset$, $\xi, \xi' \in L_W^\perp$ and $\xi \neq \xi'$ then $f$ is uniquely determined by its values on $L_W/L_P$ and*

$$\int_{\mathbb{R}^n/L_P} f(x)dx = |\det(W)| \sum_{x \in L_W/L_P} f(x).$$

# Sampling in Tomography

In tomography a function may have several variables and may be periodic in only some of them.

Let $f(x, y)$, $x \in \mathbb{R}^m$, $y \in \mathbb{R}^{n-m}$, be a function in $\mathbb{R}^n$ with periods $p_1, \ldots, p_m \in \mathbb{R}^m$ in $x$, then the function $f$ can be viewed as a function on $\mathbb{R}^m / L_P \times \mathbb{R}^{n-m}$ with Fourier transform

$$\hat{f}(k, \xi) = \frac{(2\pi)^{-\frac{(n-m)}{2}}}{|\det(P)|} \int_{\mathbb{R}^m / L_P} \int_{\mathbb{R}^{n-m}} f(x, y) e^{-i(\langle x,k \rangle + \langle y,\xi \rangle)} dydx$$

where $k \in L_P^{\perp}$ and $\xi \in \mathbb{R}^{n-m}$.

- The function $f$ must be sampled on a lattice $L_W \subseteq \mathbb{R}^n$ that has periods $p_i$, ie. $\bar{p}_i = \begin{pmatrix} p_i \\ 0 \end{pmatrix} \in L_W$.

# Sampling in Tomography

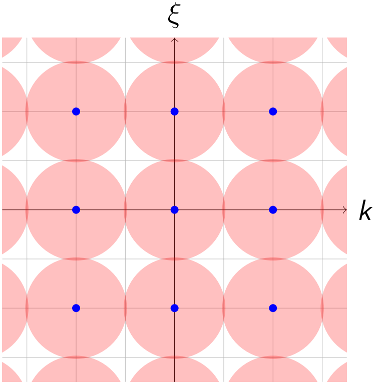
**Theorem**

*Let $f \in L_2(\mathbb{R}^m / L_P \times \mathbb{R}^{n-m})$ and let $\hat{f} = 0$ outside a set $K \subset L_P^{\perp} \times \mathbb{R}^{n-m}$. Assuming that the translates of $K$ are mutually disjoint with respect to $L_W^{\perp}$, then $f$ is uniquely determined by its values on $L_W / L_{\bar{P}}$ and*

$$\int_{\mathbb{R}^m / L_P} \int_{\mathbb{R}^{n-m}} f(x, y) dx dy = |\det(W)| \sum_{L_W / L_{\bar{P}}} f(x).$$

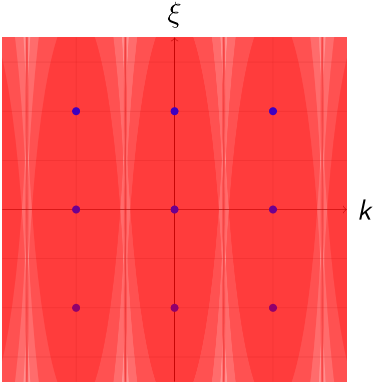
# Sampling in Tomography: Example

Let $f(x, y)$ be periodic in $x$ and bandlimited in $y$ with a circular profile in Fourier space, then this function can be sampled using the following schemes.



# Sampling in Tomography: Example

If we were to drop the bandlimit assumption on $y$ then aliasing would occur as the shifted copies of the frequency profile would overlap.



# Summary

- The theory presented so far is used in determining the efficient sampling schemes for integral transforms based on their asymptotic properties.

# Asymptotic Analysis of Integral Transforms

One approach to analyzing integral transforms (i.e. Radon, Fan-beam) involves finding an optimal sampling lattice which is determined by the asymptotic properties of the integral transform in Fourier space.

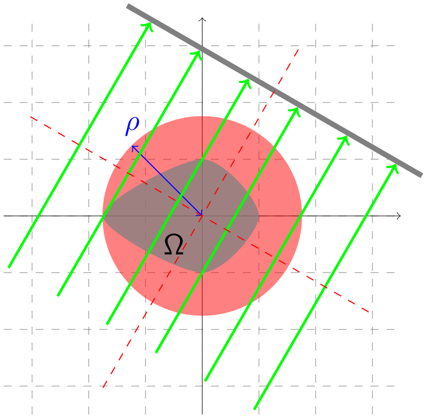
Assumptions:

- The object (function) $f$ being imaged is assumed to be supported in a ball of radius $\rho$ in $\mathbb{R}^2$ with *essential bandwidth* $\Omega$.
- There is no truncation in the projections.

## Essential Bandwidth

Let $\Omega$ be a cut-off frequency such that $|\hat{f}(\xi)|$ is sufficiently small for $|\xi| > \Omega$, then $\Omega$ may be viewed as an *essential bandwidth* of $f$ ($f$ is $\Omega$-bandlimited).

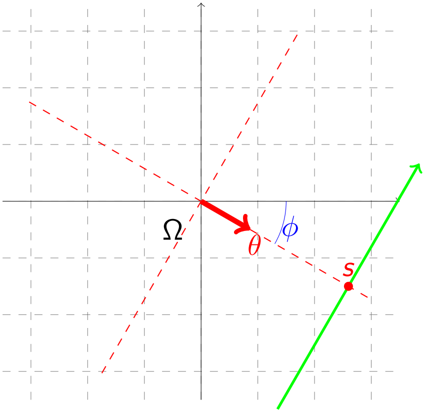
# The Radon Transform




$$\mathbf{R}f(\phi, s) = \int_{\langle x,\theta\rangle = s} f(x)dx = \int_{\vartheta^{\perp}} f(s\theta + y)dy$$

where $\vartheta^{\perp} = \{x \in \mathbb{R}^2 | \langle x, \theta\rangle = s\}$ and $\theta = (\cos\phi, \sin\phi)^{\top}$.

# The Radon Transform

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where $\vartheta^{\perp} = \{x \in \mathbb{R}^2 | \langle x, \theta\rangle = s\}$ and $\theta = (\cos\phi, \sin\phi)^{\top}$.

# Fourier Transform of Radon Transform

$$
\begin{aligned}
(\widehat{\mathbf{R}f})(k,\sigma) &= (2\pi)^{-3/2}\int_0^{2\pi}\int_{-\rho}^{\rho} e^{-i(k\phi+\sigma s)}(\mathbf{R}f)(\phi,s)\,ds\,d\phi \\
&= (2\pi)^{3/2} i^k \int_{|x|<\rho} e^{-ik\psi} f(x) J_k(-\sigma|x|)\,dx.
\end{aligned}
$$

where

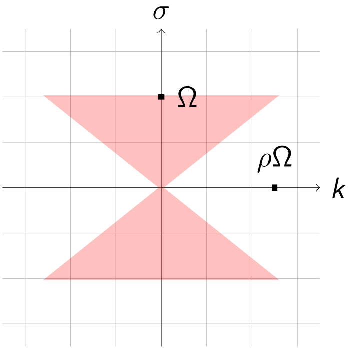
$$
J_k(-\sigma|x|) = (2\pi i^k)^{-1}\int_0^{2\pi} e^{-ik\phi - i\sigma|x|\cos\phi}\,d\phi
$$

is a Bessel function of the first kind of order $k$.

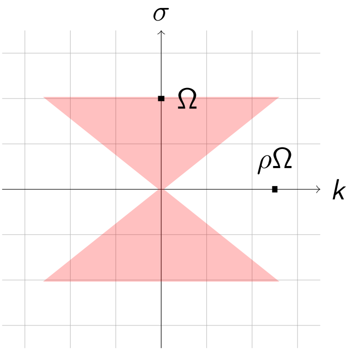
# Asymptotic Properties of the Radon Transform

Debye's asymptotic relation states that $|J_k(t)|$ decays exponentially as $|k|, |t| \to \infty$ provided that $|t| < |k|$. So $|(\widehat{\mathbf{R}f})(k,\sigma)|$ is small for $|\sigma\rho| < |k|$ (since $|x| < \rho$) which is the area outside of the set

$$K = \{(k,\sigma) | \quad |\sigma| < \Omega, \quad |\sigma\rho| < |k|\}.$$



# Asymptotic Properties of the Radon Transform

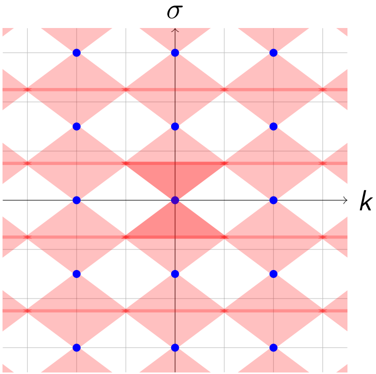



The "essential set" $K$ depicted above helps to determine

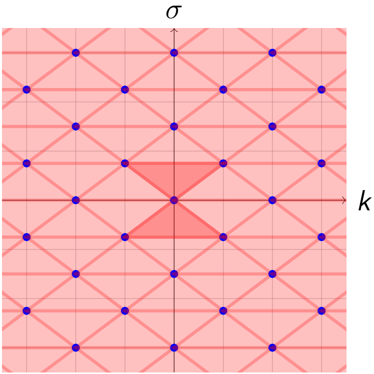
- the minimum angular distance between detector heads placed on a great circle by displacing the set along the $k$ axis, and
- the resolution of the detector by finding the minimum displacement along the $\sigma$ axis.

# Radon Transform: Optimal Sampling Lattice

An $\Omega$-bandlimited function $f \in L_2(\mathbb{R}^2)$ can be recovered exactly from its samples as long as the displacements of the set $K$ in the lattice do not overlap (see Natterer [1986]).

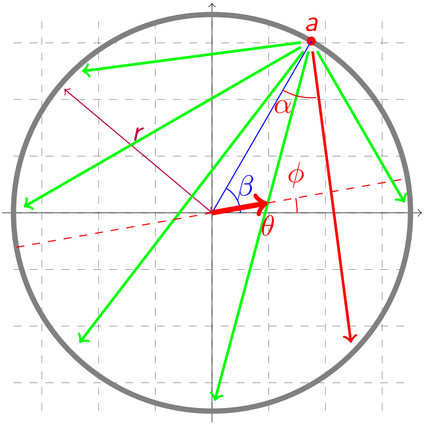



Standard Lattice




Interlaced Lattice

# The Fanbeam Transform




$$\mathbf{D}f(\beta,\alpha) = \int_{\vartheta^\perp} f(r\sin\alpha\theta + y)dt$$

where $\vartheta^\perp = \{x \in \mathbb{R}^2 \,|\, \langle x, \theta\rangle = r\sin\theta\}$, $\theta = (\cos\phi, \sin\phi)^\top$ and $\phi = \beta + \alpha - \pi/2$.

# Fourier Transform of Fanbeam Transform

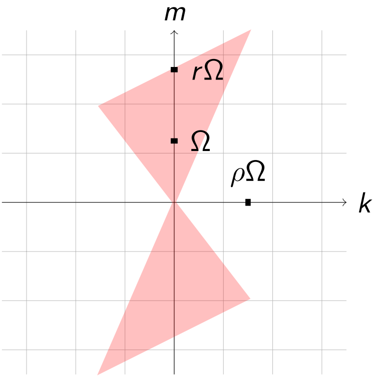
Taking the Fourier transform of the Fanbeam transform and performing some mathematical manipulation it becomes apparent that the asymptotic properties of this transform depend of two Bessel functions of the first kind.

$$\begin{aligned}\hat{g}(k,m) &= (2\pi)^{-2}\int_{\mathbb{R}^2} f(x)e^{-ik\psi}\int_{\mathbb{R}} J_k(-\sigma|x|)\int_{-\pi}^{\pi} e^{i\sigma r\sin\alpha+ik\alpha}d\alpha d\sigma dx \\ &= (2\pi)^{-1}\int_{\mathbb{R}^2} f(x)e^{-ik\psi}\int_{\mathbb{R}} J_k(-\sigma|x|)J_{m-k}(\sigma r)d\sigma dx.\end{aligned}$$

# Asymptotic Properties of the Fanbeam Transform

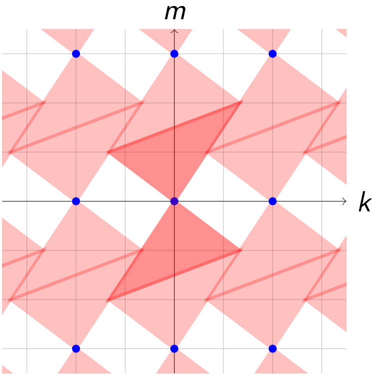
A lengthy analysis (see Palamodov [1995]; Natterer [1993]) and using Debye's asymptotic relation for both Bessel functions shows that $\hat{g}$ is small outside the set

$$K = \{(k,m) \in \mathbb{Z}^2 | \quad |k-m| < \Omega r, \quad |k| r < |k-m| \rho \}.$$

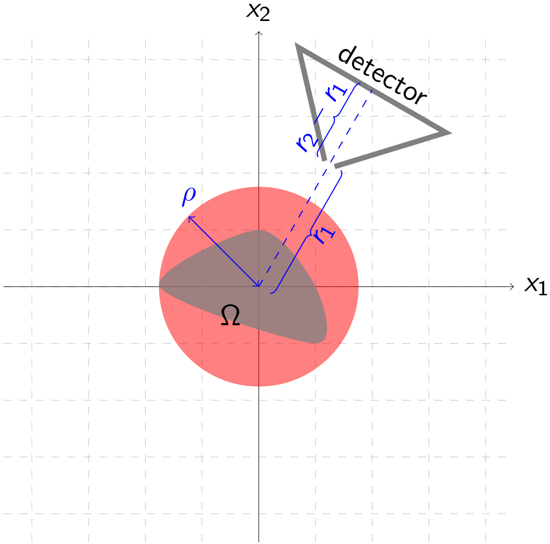


# Fanbeam Transform: Optimal Sampling Lattice

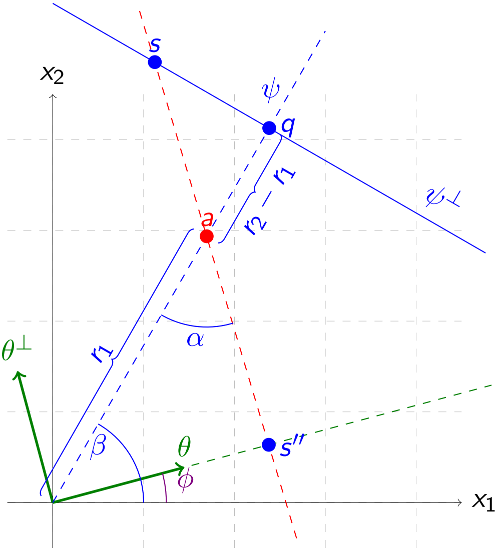
As in the case of the Radon transform, an optimal sampling lattice can be constructed dependent on the essential bandwidth Ω and the finite support of the object $\rho$, however, in this case the fan-beam radius $r$ also plays a role.



# The Single-Pinhole Transform (Flat Detector)



# The Single-Pinhole Transform (Flat Detector)



# The Single-Pinhole Transform (Flat Detector)

$$g(\beta, s) = \int_{\vartheta^\perp} f\left(r_1 \sin\left(\arctan\left(\frac{s}{r_2 - r_1}\right)\right)\theta + \ell\right) d\ell$$

where

$$\vartheta^\perp = \left\{x \in \mathbb{R}^2 | \langle x, \theta \rangle = r_1 \sin\left(\arctan\left(\frac{s}{r_2 - r_1}\right)\right)\right\},$$

$$\theta = (\cos\phi, \sin\phi)^\top$$

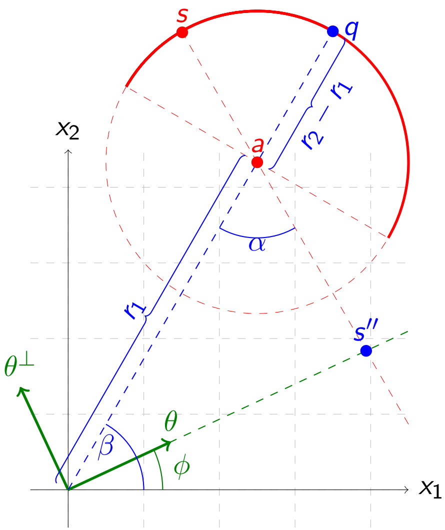
and

$$\phi = \beta + \arctan\left(\frac{s}{r_2 - r_1}\right) - \frac{\pi}{2}.$$

# The Single-Pinhole Transform (Flat Detector)

- The “flat-detector” SPT proved to be difficult to analyze.
- A “curved-detector” SPT is simpler to analyze and might bring some insight into the “flat-detector” SPT.

# Curved-Detector Single Pinhole Transform



# Curved-Detector Single Pinhole Transform

$$g(\beta, s) = \int_{\vartheta^\perp} f\left(r_1 \sin\left(\frac{s}{r_2 - r_1}\right)\theta + \ell\right) d\ell$$

where

$$\vartheta^\perp = \left\{x \in \mathbb{R}^2 | \, \langle x, \theta \rangle = r_1 \sin\left(\frac{s}{r_2 - r_1}\right)\right\},$$

$$\theta = (\cos\phi, \sin\phi)^\top$$

and

$$\phi = \beta + \frac{s}{r_2 - r_1} - \frac{\pi}{2}.$$

# Fourier Transform of the Curved Detector SPT

$$\begin{aligned}\hat{g}(k,\sigma) &= \frac{1}{2\pi^2}\int_{-\pi}^{\pi}\int_{-\frac{\pi}{2}(r_2-r_1)}^{\frac{\pi}{2}(r_2-r_1)} g(\beta,s)e^{-ik\beta-i\sigma s}dsd\beta \\ &= \frac{1}{\pi}\int_{\mathbb{R}^2} f(x)e^{-ik\gamma}\int_{\mathbb{R}} J_k(-\varsigma|x|)I_{k-(r_2-r_1)\sigma}(\varsigma r_1)d\varsigma dx\end{aligned}$$

where

$$I_\ell(\varsigma r_1) = \int_{-\frac{\pi}{2}(r_2-r_1)}^{\frac{\pi}{2}(r_2-r_1)} e^{i\varsigma r_1 \sin(\frac{s}{r_2-r_1})+i\ell\frac{s}{r_2-r_1}} ds.$$

and $J_k(-\varsigma|x|)$ is a Bessel function of the first kind of order $k$.

# Asymptotic Properties of the Curved-Detector SPT

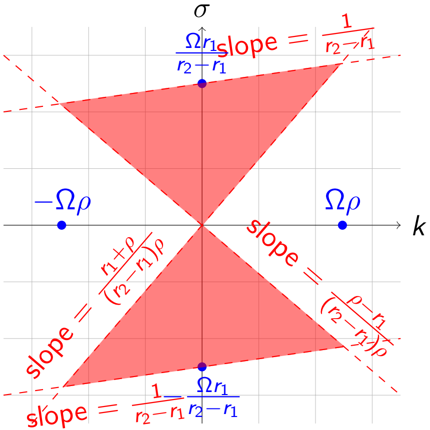
- The asymptotic properties of $J_k(-\varsigma|x|)$ are known due to Debye's asymptotic relation.
- The asymptotic properties of $I_\ell(\varsigma r_1)$ were found using *stationary phase approximation* employing the techniques in (Bleistein and Handelsman [1975]).

The resulting "essential set" is

$$K = \{(k,\sigma)| \quad |k-\sigma(r_2-r_1)| < \Omega r_1, |k|r_1 < |k-\sigma(r_2-r_1)|\rho\}.$$

# Asymptotic Properties of the Curved-Detector SPT

$$K = \{(k, \sigma) | \quad |k - \sigma(r_2 - r_1)| < \Omega r_1, |k| r_1 < |k - \sigma(r_2 - r_1)|\rho\}.$$



# Summary

- Using stationary phase approximation the asymptotic properties of the curved-detector SPT were found.
- The essential set $K$ of the curved-detector SPT is a re-parameterization of the fan-beam transform.
  - $\Rightarrow$ The intent was to find a theoretical optimum for the design of a SPECT device which can be used as a starting point in numerical methods that account for additional factors.

Thank you

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