$$\sum_{l\in \mathbf{Z}^n} \hat{f}\left(\xi - 2\pi W^{-t} l\right) = \frac{1}{\sqrt{2\pi}^n} \left|\det W\right| \sum_{k\in \mathbf{Z}^n} f(Wk) e^{-i\xi . Wk}$$

# Sampling
# In
# Tomography

$$Rf(\theta, s) = \int_{y\in\theta^\perp} f(s\theta + y)\,dy$$


Laurent Desbat

TIMC – IMAG, UJF, Grenoble


Santiago 2003

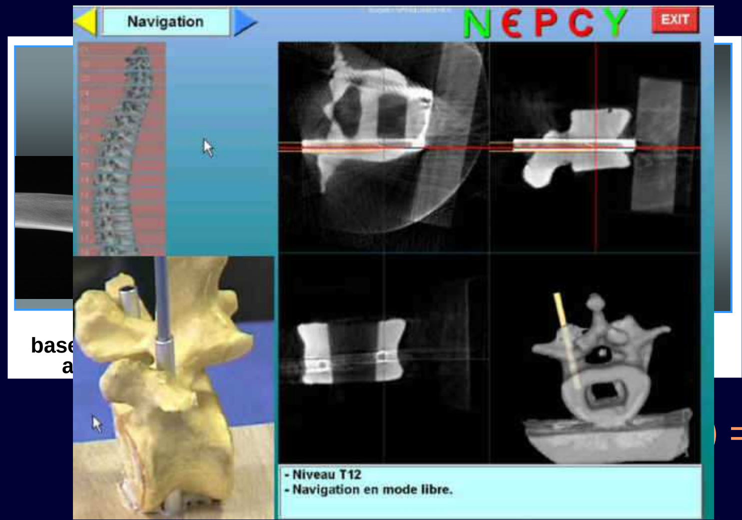
# The MI3 project

## Minimal Invasive Interventional Imaging

Jan. 2000 - Dec. 2002
Budget : EUR. 2.664.000

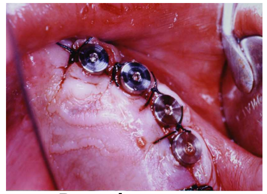
D W A R E

3D scanner

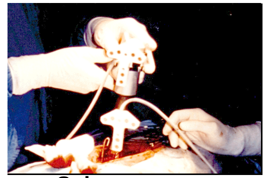



$$= \int_{-\infty}^{\infty} f(x+t\zeta)dt$$

C L I N I C A L



Dental surgery



Spine surgery

A P P L I C A T I O N S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A project supported by

The European Commission
Community Research
ist information society technologies

Project number : IST-1999-12338
Key action : Systems and services for the citizen
Action line : Health

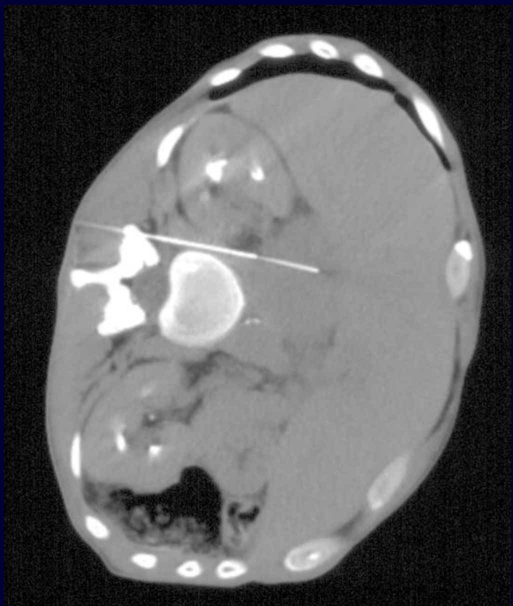

http://mi3.vitamib.com/

# DynCT : 3D TOMO-FLUOROSCOPY
# Image Guided Intervention

DynCT

IST 2000-2003

*Low contrast region*



*Radiology, Mai 1999*

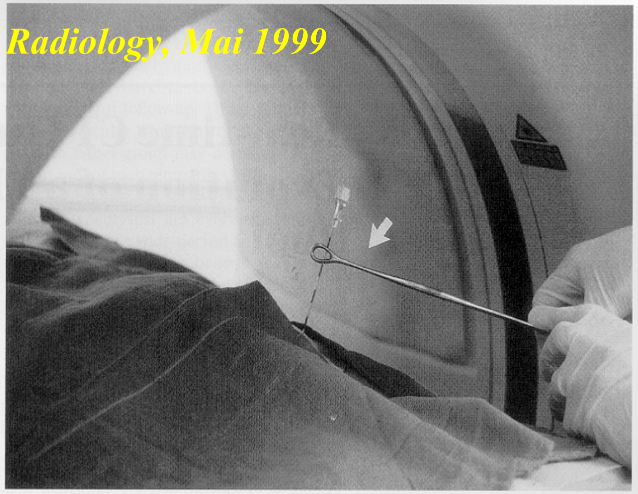


Figure 1. For biopsy or for access needle placement under continuous CT fluoroscopic guidance, a stainless steel sponge forceps (arrow) is currently our needle immobilization device of choice. This reduces secondary radiation scatter to the operator's hands.

*Image Guided Biopsy*

*Bad resolution and artifacts due to organ movements*

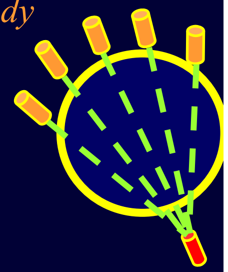

$$g(\phi,s)=\int_{x.\theta=s} f(D_\phi(x))dx$$

# Intra-operative Tomography

- Fast reconstruction methods [T. Rodet]
- Dynamic tomography [S. Roux]
- New identification approaches: local approaches and model driven acquisition [M. Fleute, A. Bilgot]
- Fast and efficient acquisition systems

➔**Sampling** $\sum_{l \in \mathbf{Z}^n} \hat{f}\left(\xi - 2\pi W^{-t} l\right) = \frac{1}{\sqrt{2\pi}^n} \left|\det W\right| \sum_{k \in \mathbf{Z}^n} f(Wk) e^{-i\xi . Wk}$

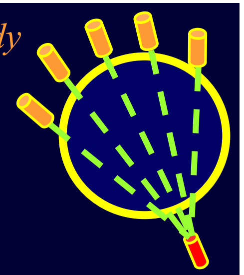
$Rf(\theta,s)=\int_{y\in\theta^{\perp}} f(s\theta+y)dy$



# Plan

- Tomography
- Sampling
- Sampling in tomography (Shannon approach)
- Sampling the Fan Beam X-ray Transform
- Efficient sampling in parallel helical tomography
- Discussion and perspectives

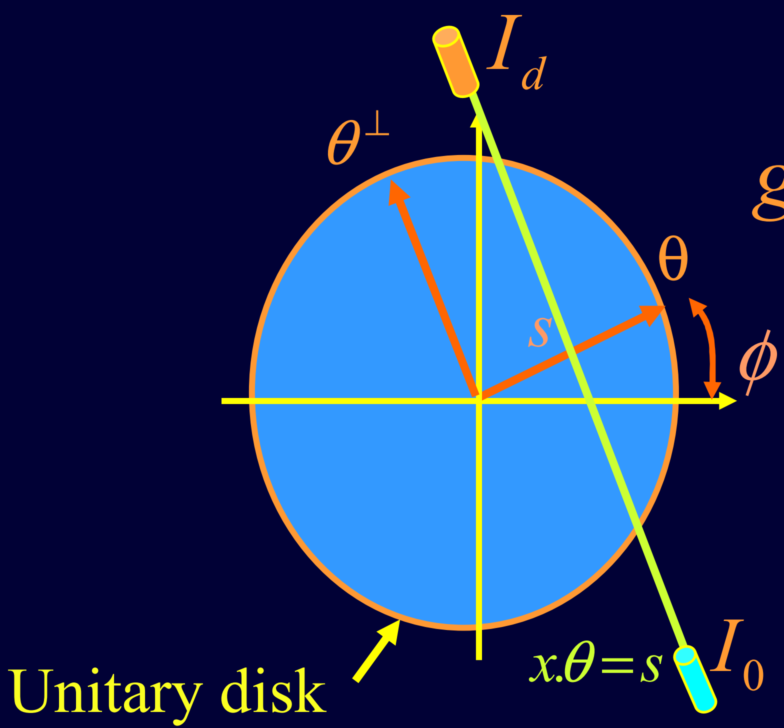
$$Rf(\theta,s)=\int_{y\in\theta^{\perp}} f(s\theta+y)\,dy$$



- Radon Transform

$f \in C_0^{\infty}\left(\mathbf{R}^2\right)$ attenuation function

$$\frac{I_d}{I_0} = e^{-\int_{x.\theta=s} f(x)\,dx}$$




$$g(\phi,s) = Rf(\phi,s) = \int_{x.\theta=s} f(x)\,dx$$

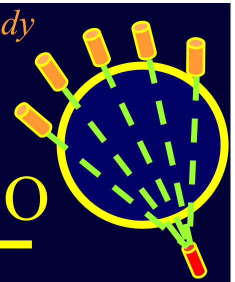
$$Rf(\theta,s)=\int_{y\in\theta^{\perp}} f(s\theta+y)\,dy$$

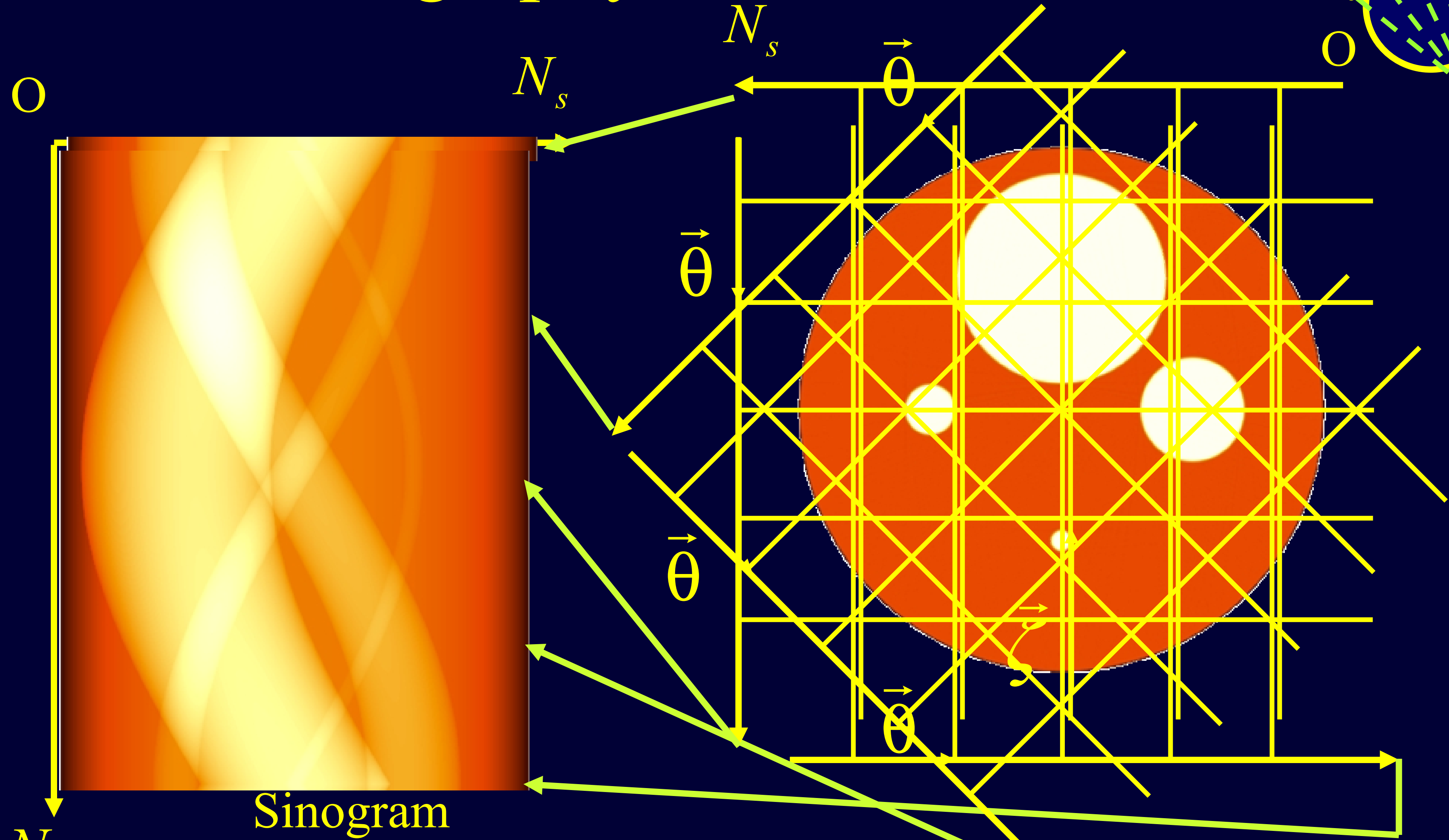
$$Z=\left\{(\theta,s),\theta\in\mathbf{S}^{n-1},s\in\mathbf{R}\right\}$$

$$\left(R_{\theta}f\right)(s)=\left(Rf\right)(\theta,s)$$

# 2D Tomography: Radon Transform

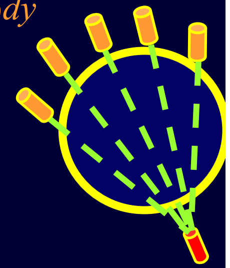
$$Rf(\theta,s) = \int_{y\in\theta^{\perp}} f(s\theta + y)dy$$





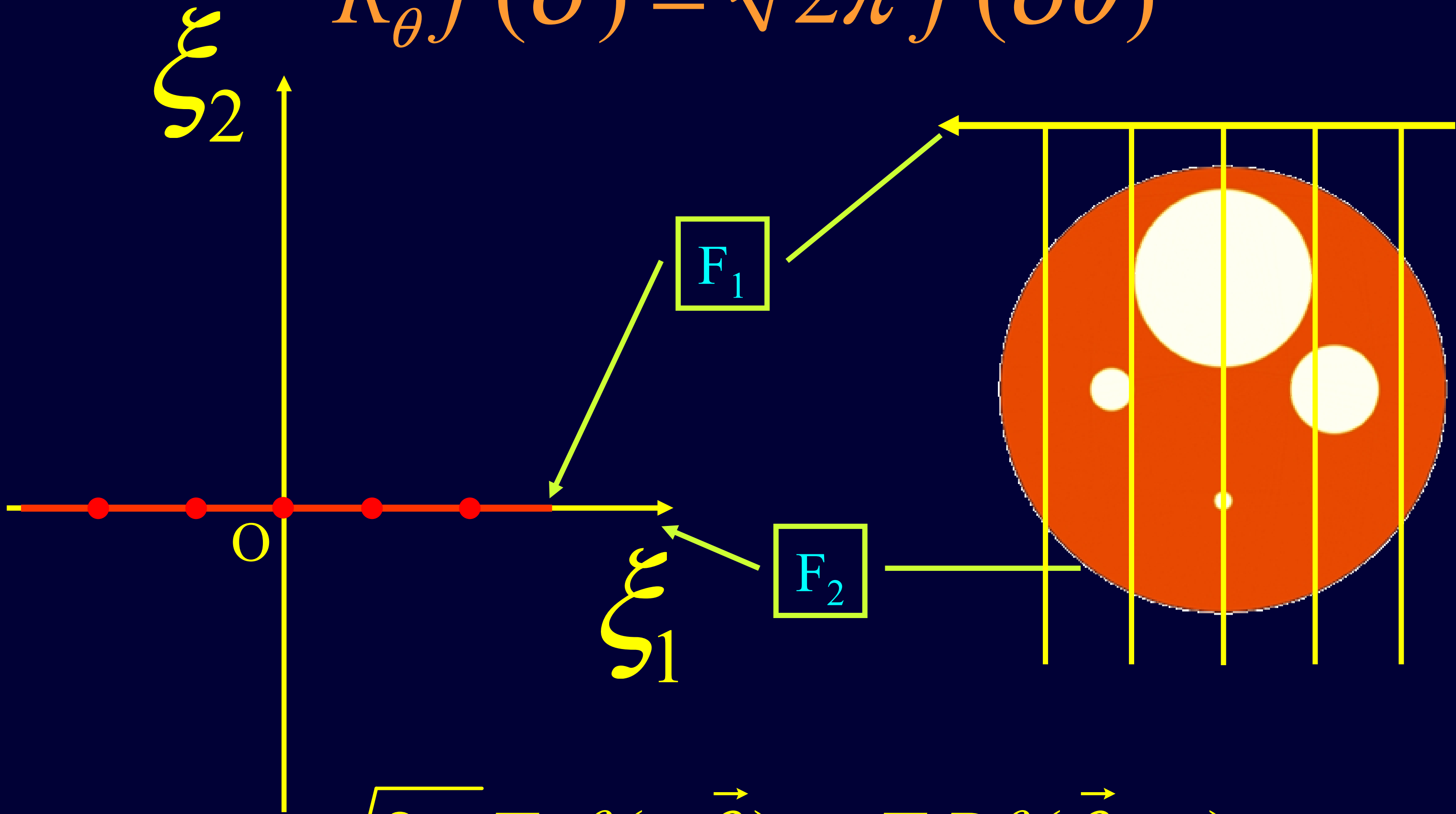

$$\vec{\theta} = \begin{pmatrix} \cos(\phi) \\ \sin(\varphi) \end{pmatrix}, \mathrm{R}f\ (\varphi, s) = \mathrm{R}_{\phi} f\ (s) = \int_{\Re} f\ (s\vec{\theta} + t\vec{\zeta})dt$$

$$Rf(\theta,s)=\int_{y\in\theta^{\perp}} f(s\theta+y)\,dy$$



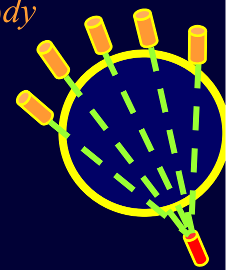
# Projection Slice Theorem

$$\widehat{R_\theta f}(\sigma)=\sqrt{2\pi}\,\hat{f}(\sigma\theta)$$




$$\sqrt{2\pi}F_2 f(\sigma\vec{\theta})=F_1 Rf(\vec{\theta},\sigma)$$

$$Rf(\theta,s)=\int_{y\in\theta^{\perp}} f(s\theta+y)\,dy$$



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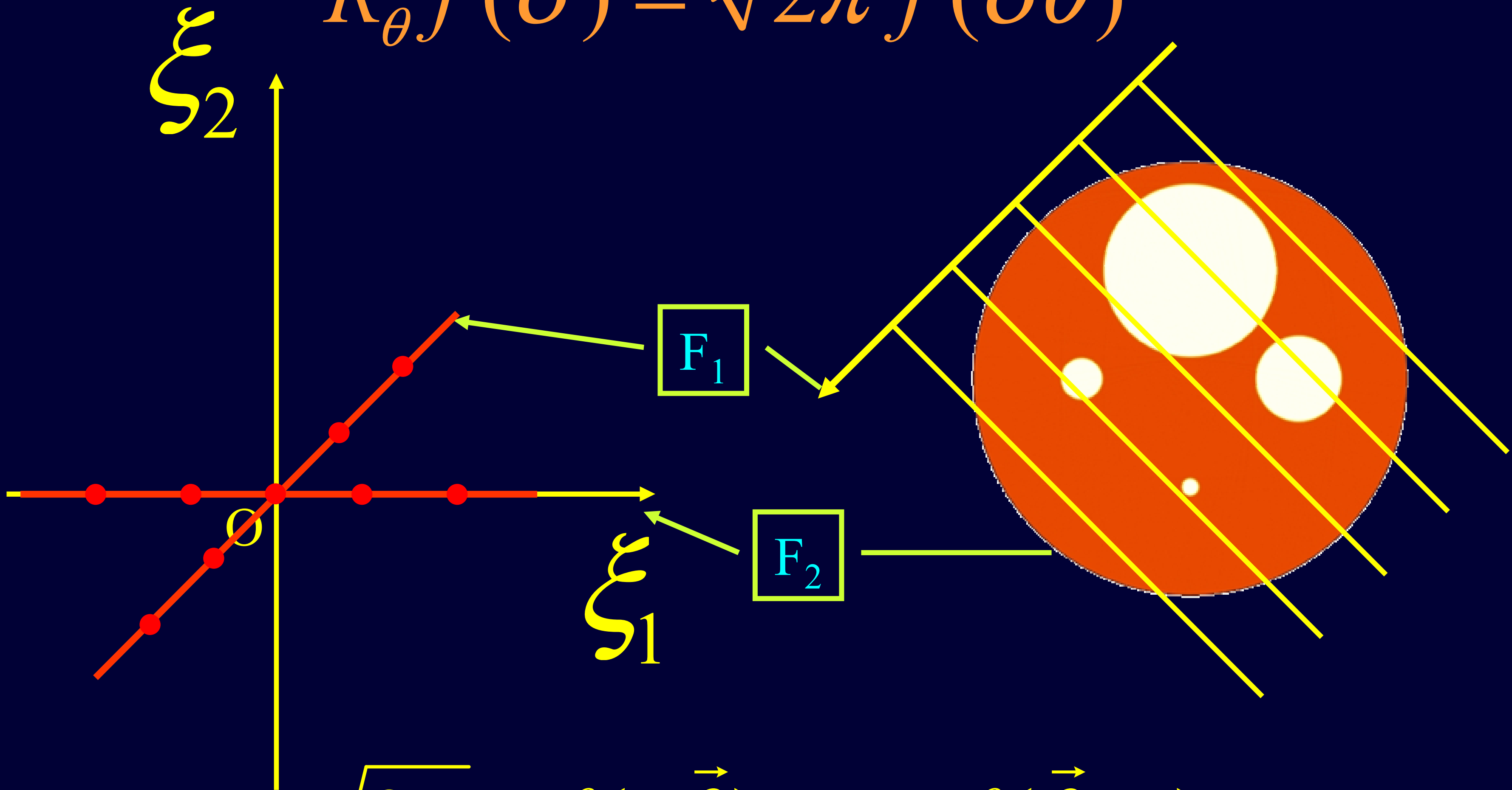


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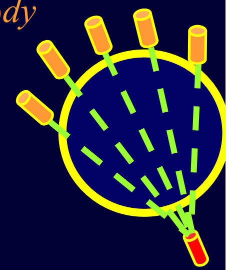

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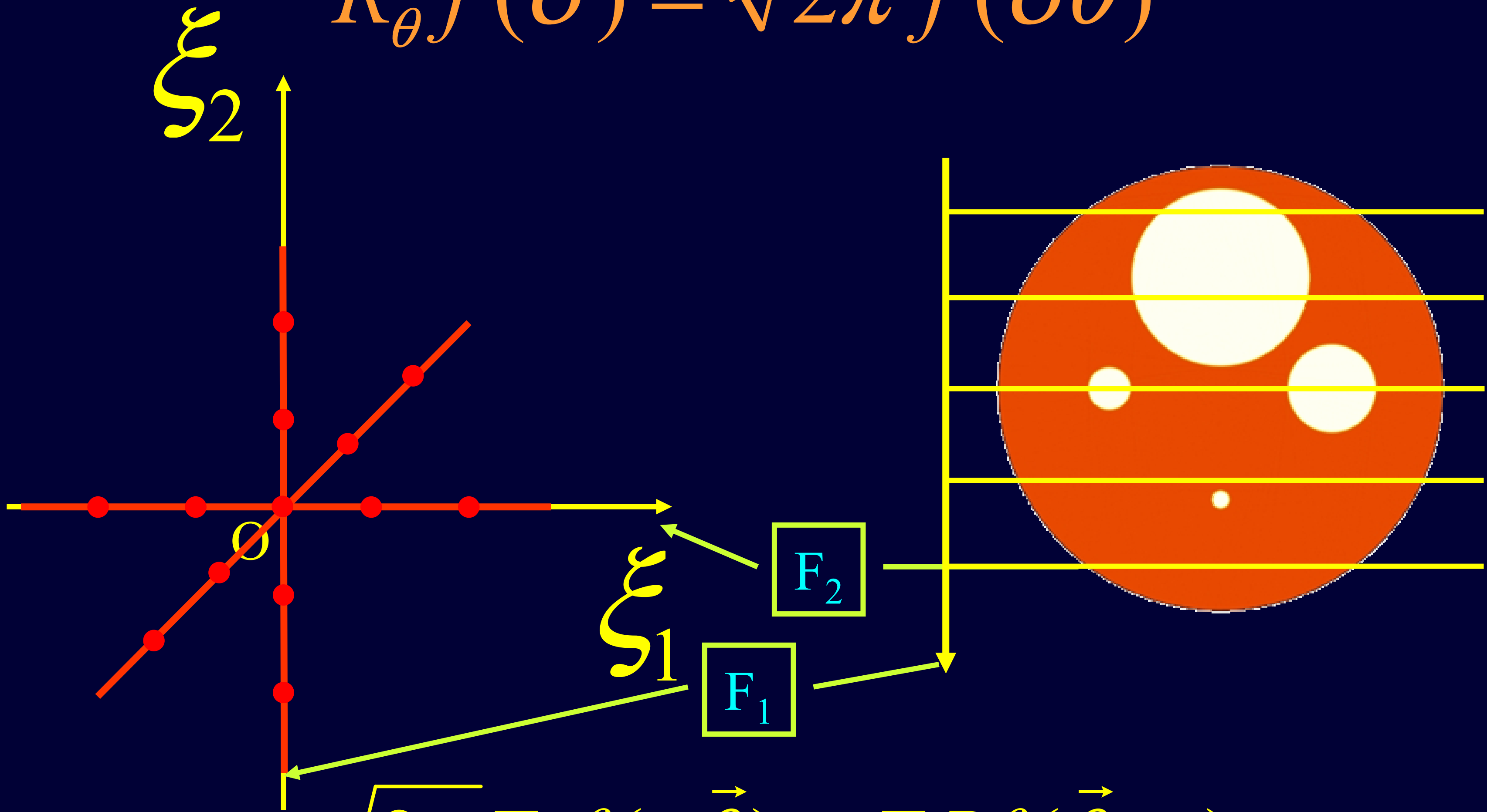


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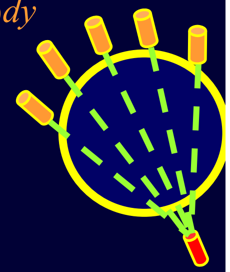

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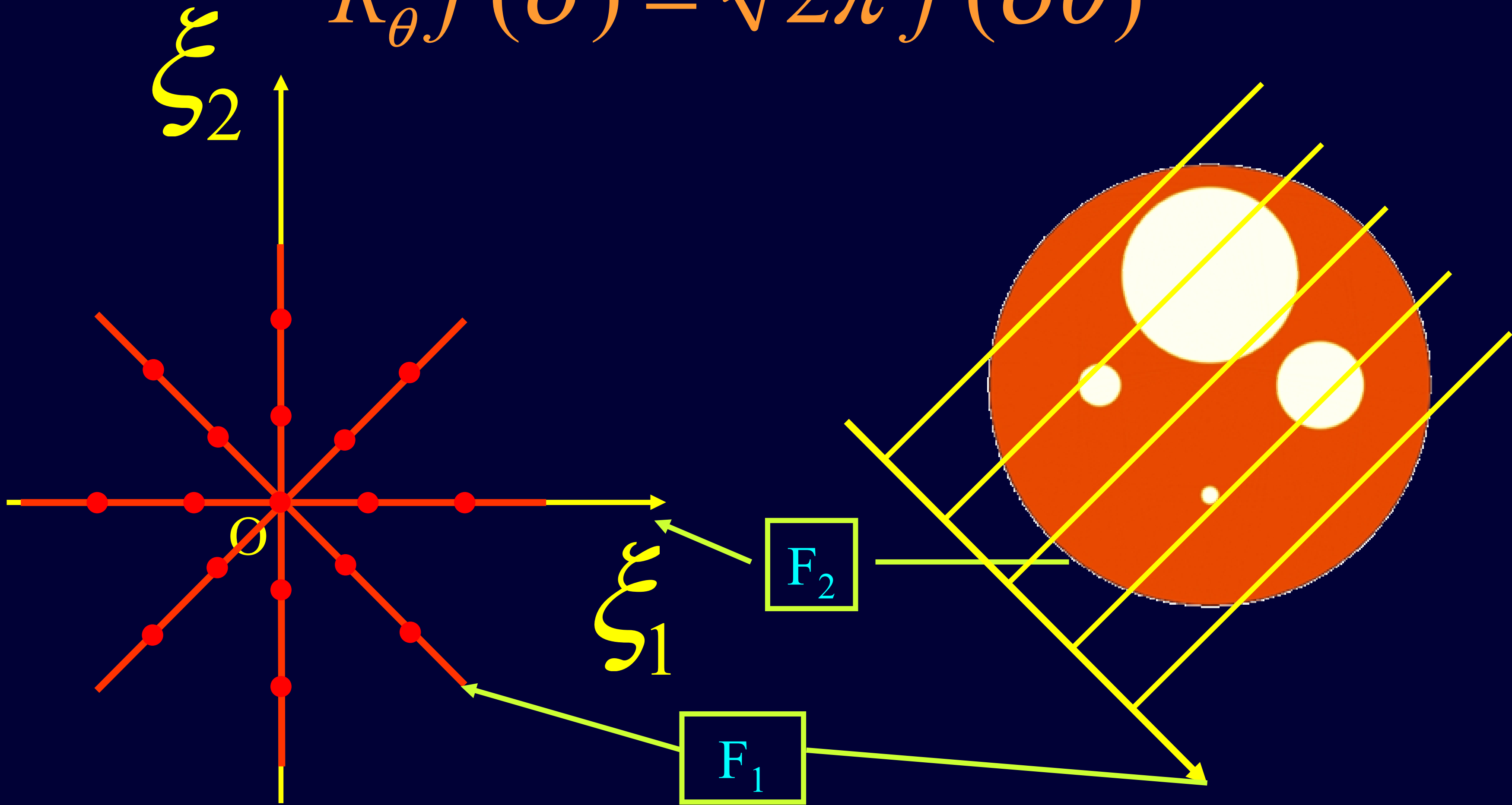


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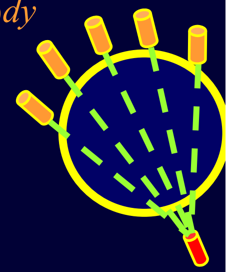

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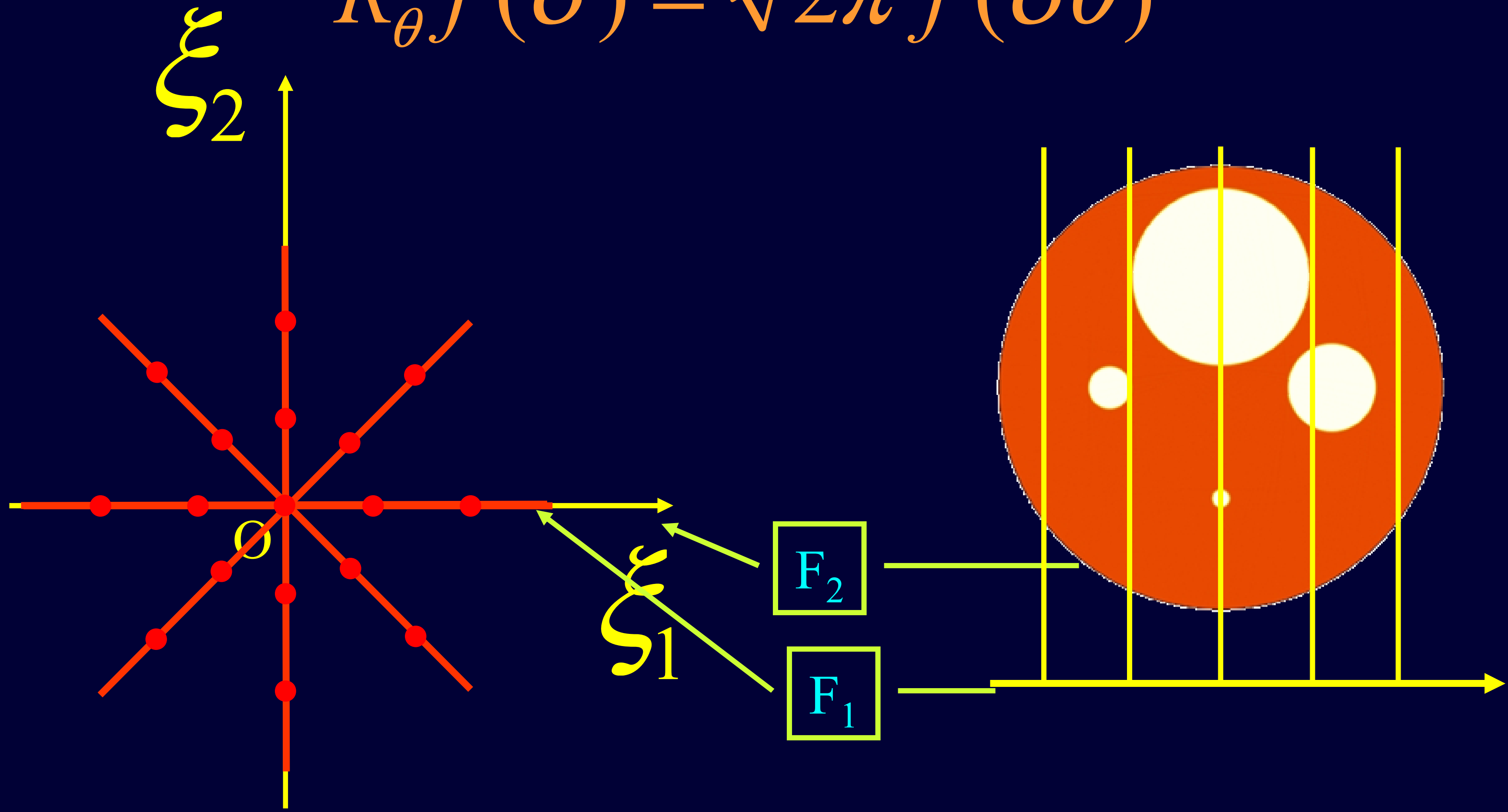


$$\sqrt{2\pi}F_2 f(\sigma\vec{\theta})=F_1 Rf(\vec{\theta},\sigma)$$

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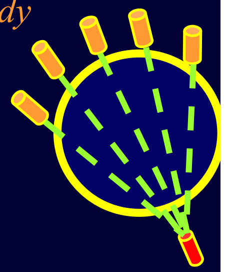


# Inversion of the Radon Transform

- Inversion formula:

$$f=\frac{1}{2}\,R^{\#}I^{-1}Rf$$

- Filtering:

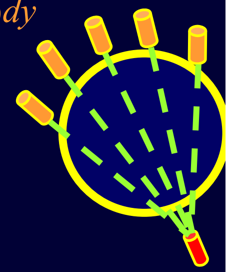
$$I^{-1}Rf\,(\vec{\theta},s)=\frac{1}{2\pi}\int_{\Re}|\sigma|\mathrm{F}_1 Rf\,(\vec{\theta},\sigma)\,e^{i\sigma s}\,ds$$

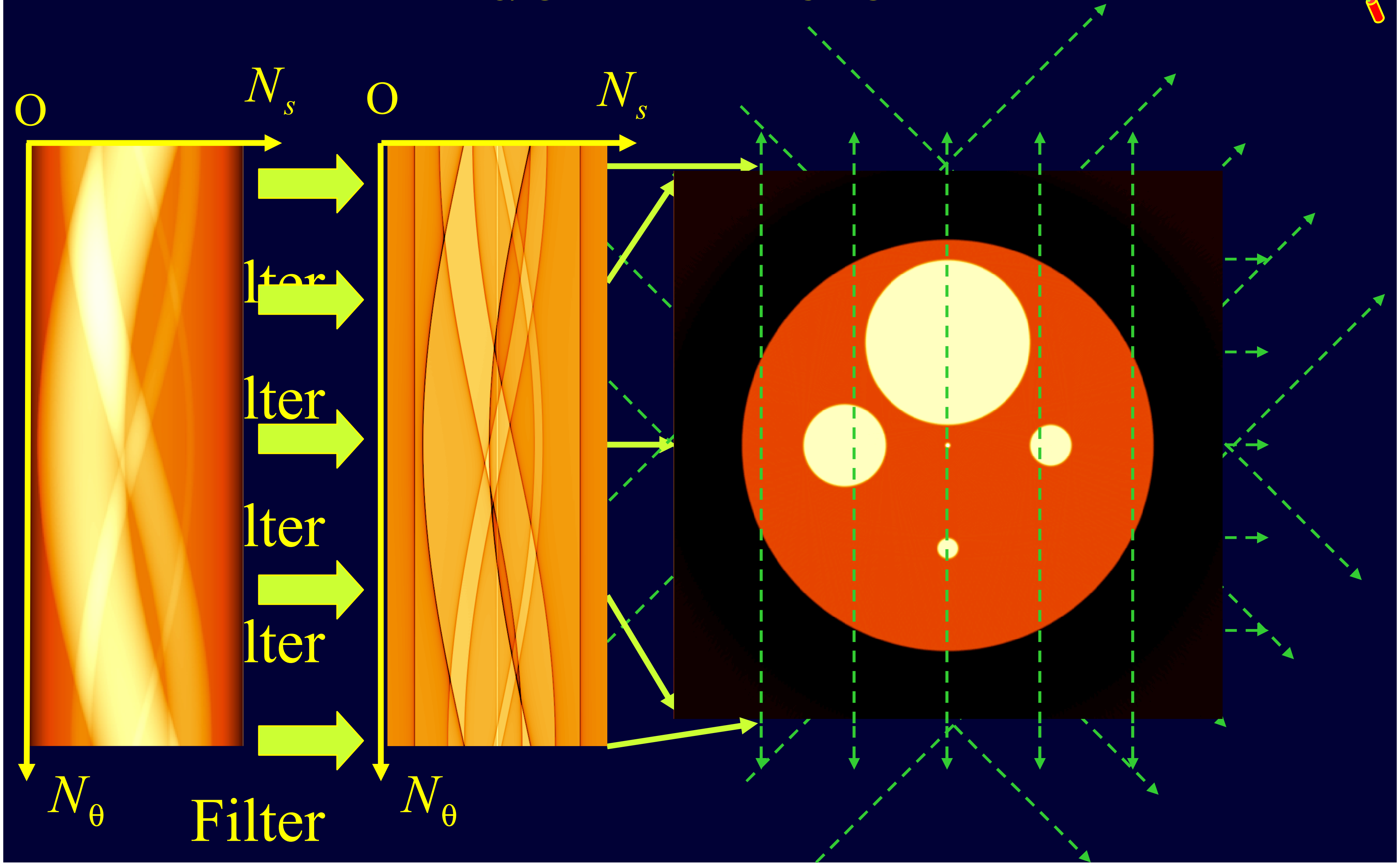
- Backprojection:

$$\mathrm{R}^{\#}g\,(x)=\int_{\vec{\theta}\in S^1} g(\vec{\theta},x\cdot\vec{\theta})d\vec{\theta}$$

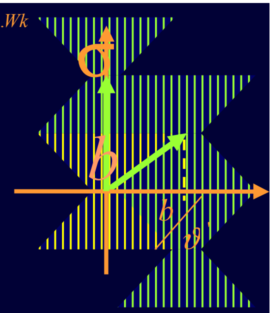
# Inversion of the Radon Transform

$$Rf(\theta,s) = \int_{y\in\theta^{\perp}} f(s\theta + y)\,dy$$





$$\sum_{l\in\mathbf{Z}^n} \hat{f}\left(\xi - 2\pi W^{-t} l\right) = \frac{1}{\sqrt{2\pi}^n} \left|\det W\right| \sum_{k\in\mathbf{Z}^n} f(Wk) e^{-i\xi . Wk}$$



# Plan

- Tomography
- Sampling
- Sampling in tomography (Shannon approach)
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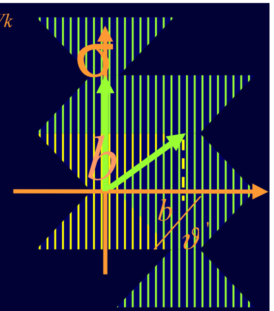


- Definition

$$n \in \mathbf{N}, f \in \mathbf{L}^1(\mathbf{R}^n),$$

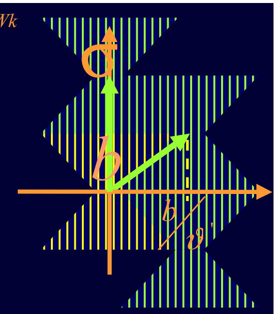
  - Fourier Transform

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} f(x) e^{-ix.\xi} dx$$

  - If $f$ is continue and $\hat{f} \in \mathbf{L}^1(\mathbf{R}^n)$,

$$f(x) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} \hat{f}(\xi) e^{ix.\xi} d\xi$$

$$\sum_{l\in\mathbf{Z}^n}\hat{f}\left(\xi-2\pi W^{-t}l\right)=\frac{1}{\sqrt{2\pi}^n}\left|\det W\right|\sum_{k\in\mathbf{Z}^n}f(Wk)e^{-i\xi.Wk}$$

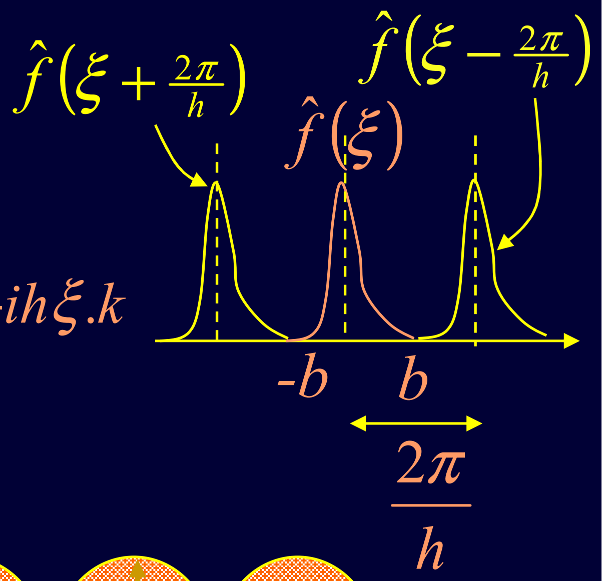


- Poisson Formula

Let $f$ be a function to sample,

$$f\in\mathbf{L}^1(\mathbf{R}^n)\bigcap\mathbf{L}^2(\mathbf{R}^n),\,h>0$$

$$\sum_{l\in\mathbf{Z}^n}\hat{f}\left(\xi-\frac{2\pi}{h}l\right)=\frac{1}{\sqrt{2\pi}^n}h^n\sum_{k\in\mathbf{Z}^n}f(hk)e^{-ih\xi.k}$$

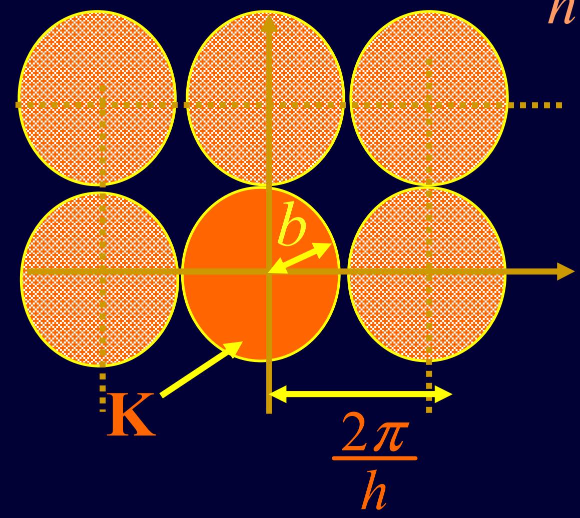



Let **K** be a set containing the support of $\hat{f}(\xi)$

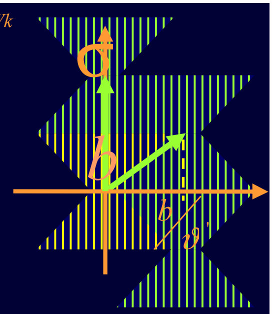
Choose $h$ such that the sets:

$\mathbf{K}+\frac{2\pi}{h}\mathbf{Z}^n$ do not overlapp

$\frac{\pi}{h}\geq b$ Shannon Condition

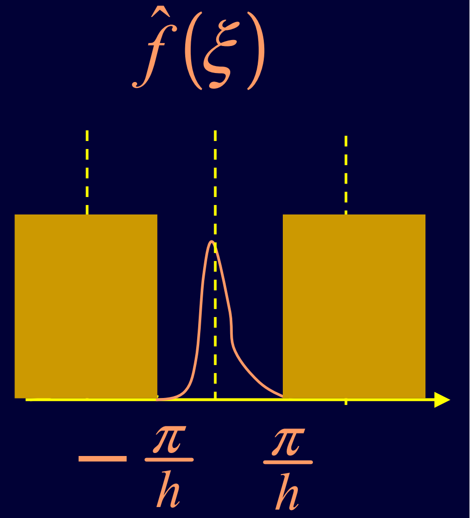


$$\sum_{l\in \mathbf{Z}^n} \hat{f}\left(\xi - 2\pi W^{-t} l\right) = \frac{1}{\sqrt{2\pi}^n} \left|\det W\right| \sum_{k\in \mathbf{Z}^n} f(Wk) e^{-i\xi.Wk}$$



- Sampling formula

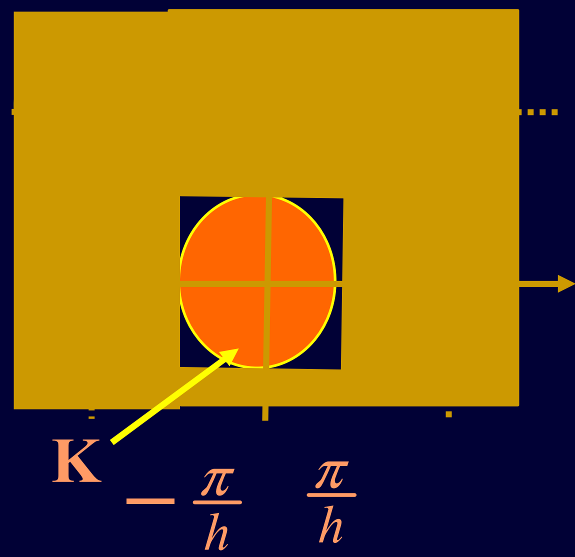
Let $f$ be a function to be sampled



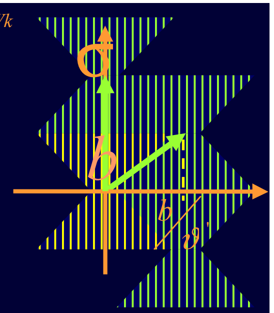

$$\sum_{l\in \mathbf{Z}^n} \hat{f}\left(\xi - \frac{2\pi}{h} l\right) = \frac{1}{\sqrt{2\pi}^n} h^n \sum_{k\in \mathbf{Z}^n} f(hk) e^{-ih\xi.k}$$

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}^n} h^n \sum_{k\in \mathbf{Z}^n} f(hk) e^{-ih\xi.k} \chi_{\left[-\frac{\pi}{h}, \frac{\pi}{h}\right]^n}(\xi)$$

$$f(x) = \sum_{k\in \mathbf{Z}^n} f(hk) \,\mathrm{sinc}\left(\tfrac{\pi}{h}(x - hk)\right)$$



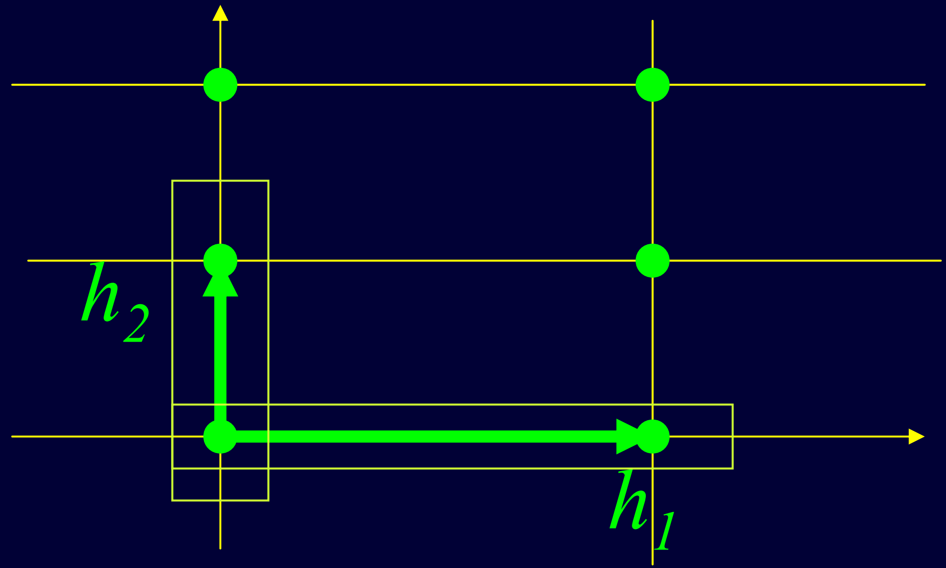
$$\sum_{l\in \mathbf{Z}^n} \hat{f}\left(\xi - 2\pi W^{-t} l\right) = \frac{1}{\sqrt{2\pi}^n}\left|\det W\right| \sum_{k\in \mathbf{Z}^n} f(Wk) e^{-i\xi . Wk}$$



# Sampling schemes

- Equidistant on each variable: standard schemes

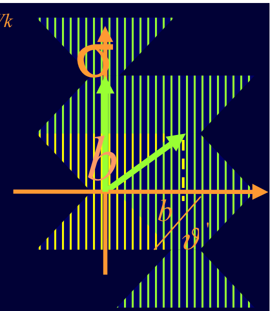
$$W = \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix}$$




$$f(k_1 h_1, k_2 h_2), k = (k_1, k_2) \in Z^2$$

$$f(Wk), k \in Z^2$$

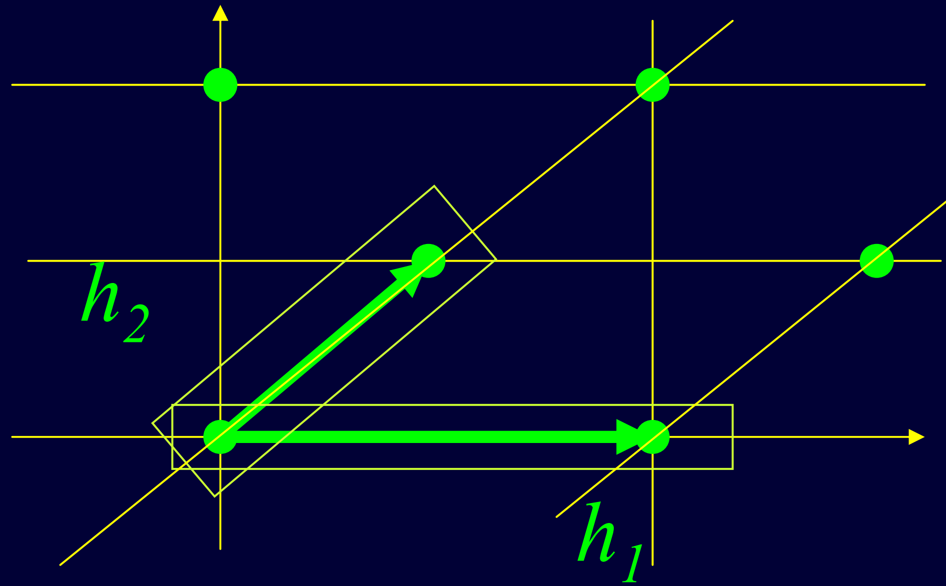
$$\sum_{l\in \mathbf{Z}^n} \hat{f}\left(\xi - 2\pi W^{-t} l\right) = \frac{1}{\sqrt{2\pi}^n} \left|\det W\right| \sum_{k\in \mathbf{Z}^n} f(Wk) e^{-i\xi . Wk}$$



# Sampling lattices

- Equidistant on independent vectors

$$W = \begin{bmatrix} h_1 & \frac{h_1}{2} \\ 0 & h_2 \end{bmatrix}$$

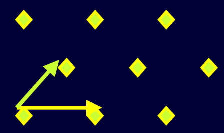


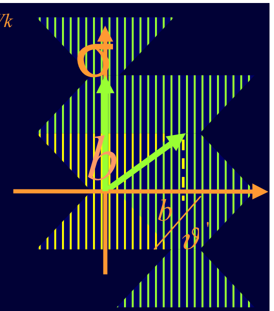

$$f(Wk), k \in Z^2$$

$\sum_{l\in\mathbf{Z}^n}\hat f\left(\xi-2\pi W^{-t}l\right)=\frac{1}{\sqrt{2\pi}^n}\left|\det W\right|\sum_{k\in\mathbf{Z}^n} f(Wk)e^{-i\xi.Wk}$

- Sampling on lattices

$$W=\begin{bmatrix} w_{1,1}\,w_{1,2}\\ w_{2,1}\,w_{2,2}\end{bmatrix}$$





Let $W$ be a non singular matrix

Poisson formula for $f$ and $W$

$$\sum_{l\in\mathbf{Z}^n}\hat f\left(\xi-2\pi W^{-t}l\right)=\frac{1}{\sqrt{2\pi}^n}\left|\det W\right|\sum_{k\in\mathbf{Z}^n} f(Wk)e^{-i\xi.Wk}$$

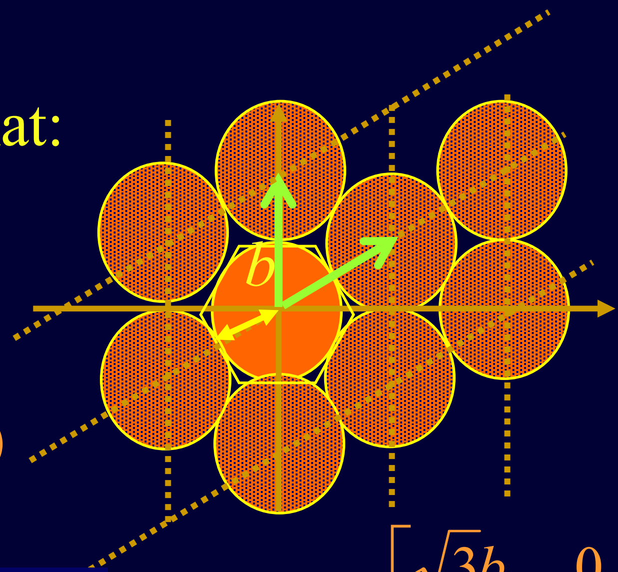
essential

$\mathbf{K}$ is the support of $\hat f$ ; $W$ is chosen such that:

$\mathbf{K}+2\pi W^{-t}\mathbf{Z}^n$ are disjoints sets.

Shannon

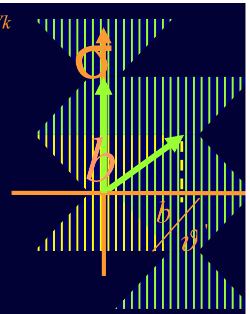
$$S_W f(x)=\frac{1}{\sqrt{2\pi}^n}\left|\det W\right|\sum_{k\in\mathbf{Z}^n} f(Wk)\hat\chi_{\mathbf{K}}(Wk-x)$$




$$2\pi W_H^{-t}=\begin{bmatrix}\sqrt{3}b & 0\\ b & 2b\end{bmatrix}$$

$$\left\|S_W f-f\right\|_\infty\le 2(2\pi)^{-n/2}\int_{\mathbf{R}^n\setminus\mathbf{K}}\left|\hat f(\xi)\right|d\xi$$

$$\sum_{l\in \mathbf{Z}^n} \hat{f}(\xi - 2\pi W^{-t} l) = \frac{1}{\sqrt{2\pi}^n} |\det W| \sum_{k\in \mathbf{Z}^n} f(Wk) e^{-i\xi . Wk}$$



# Efficient sampling

Among all matrices $W$ satisfying Shannon :

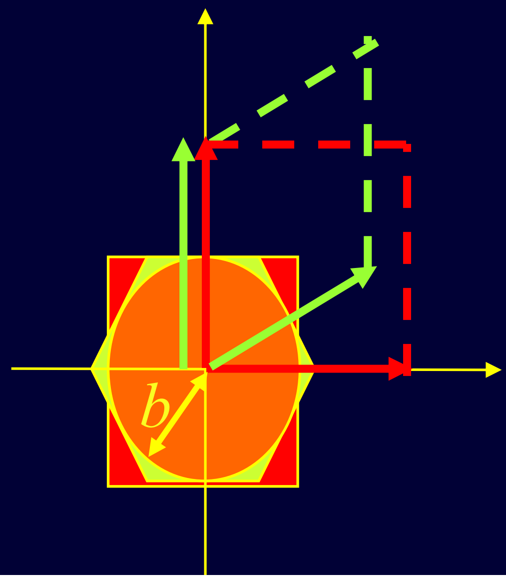
$\mathbf{K} + 2\pi W^{-t}\mathbf{Z}^n$ 2 by 2 disjoint sets

we search for those maximizing the elementary mesh area in the direct space $|\det W|$, i.e. minimizing the number of sampling points per unit area, or equivalently maximizing $|\det W^{-t}|$

$$|\det W_S| < |\det W_H|$$

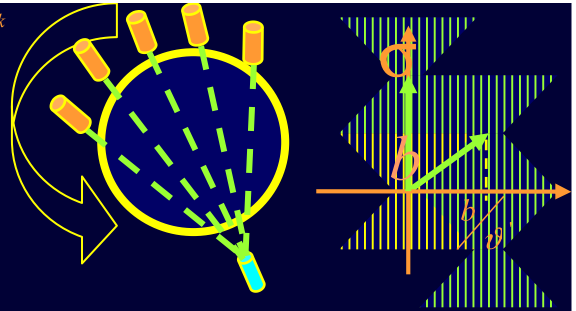
$$|\det W_H^{-t}| < |\det W_S^{-t}|$$

$$2\pi W_H^{-t} = \begin{bmatrix} \sqrt{3}b & 0 \\ b & 2b \end{bmatrix} \quad 2\pi W_S^{-t} = \begin{bmatrix} 2b & 0 \\ 0 & 2b \end{bmatrix}$$



$$\sum_{l\in\mathbf{Z}^n}\hat{f}\left(\xi-2\pi W^{-t}l\right)=\frac{1}{\sqrt{2\pi}^n}\left|\det W\right|\sum_{k\in\mathbf{Z}^n} f(Wk)e^{-i\xi.Wk}$$

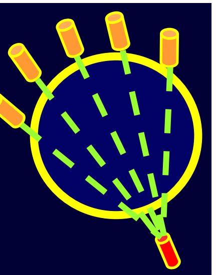
$$Rf(\theta,s)=\int_{y\in\theta^\perp} f(s\theta+y)dy$$



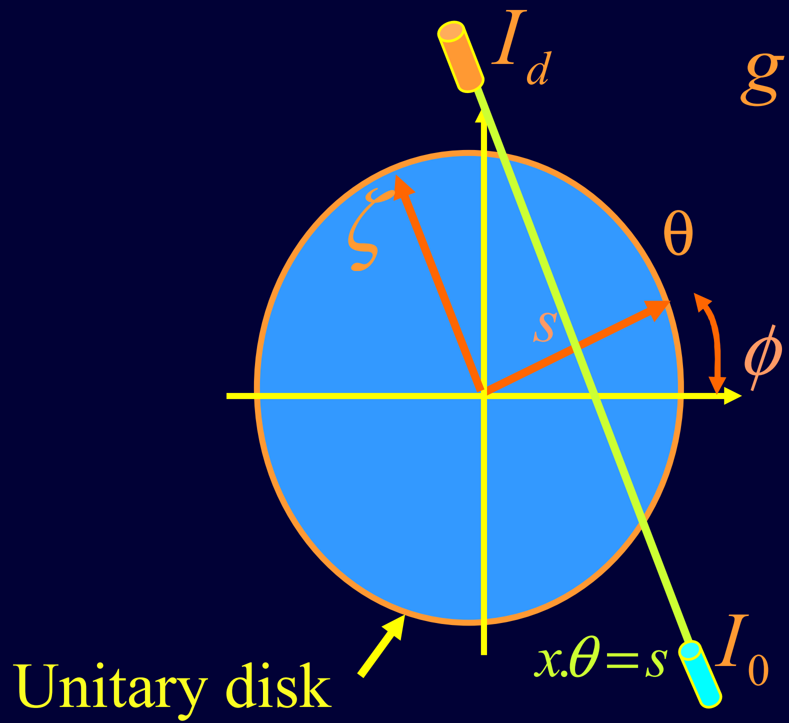
# Plan

- Tomography
- Sampling
- Sampling in tomography (Shannon approach)
- Sampling the Fan Beam X-ray Transform
- Efficient sampling in parallel helical tomography
- Discussion and perspective

$$Rf(\theta,s) = \int_{y\in\theta^{\perp}} f(s\theta + y)dy$$



- We want to sample the Radon transform




$$g(\phi,s) = Rf(\phi,s) = \int_{x.\theta=s} f(x)dx$$

$$Rf(\theta,s) = \int_{y\in\theta^{\perp}} f(s\theta + y)dy$$

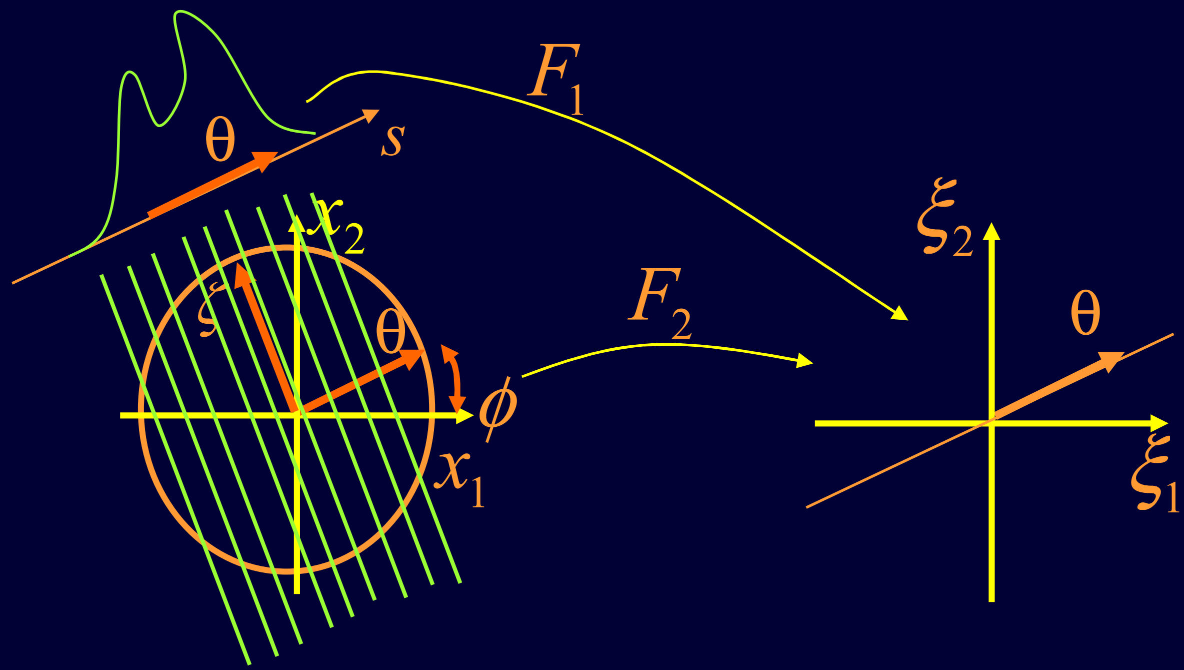
$$Z = \{(\theta,s), \theta \in \mathbf{S}^{n-1}, s \in \mathbf{R}\}$$

$$(R_\theta f)(s) = (Rf)(\theta,s)$$

$$Rf(\theta,s)=\int_{y\in\theta^{\perp}} f(s\theta+y)\,dy$$

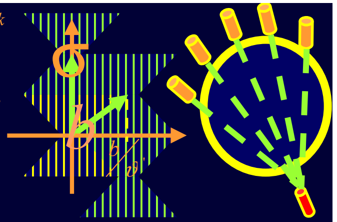
# Projection Slice Theorem

$$\widehat{R_\theta f}(\sigma)=\sqrt{2\pi}\,\hat{f}(\sigma\theta)$$



$$\sum_{l\in \mathbf{Z}^n} \hat f(\xi - 2\pi W^{-t} l) = \frac{1}{\sqrt{2\pi}^n} |\det W| \sum_{k\in \mathbf{Z}^n} f(Wk) e^{-i\xi . Wk}$$

$$Rf(\theta,s) = \int_{y\in\theta^\perp} f(s\theta + y)\,dy$$

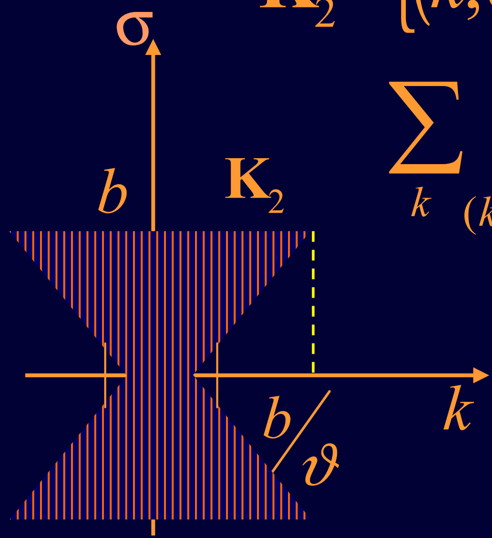


In tomography, we want to sample $g(\phi,s)=Rf(\phi,s)$

$$\hat g(\phi,\sigma) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-is\sigma} g(\phi,s)\,ds \qquad \hat g_k(\sigma) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\phi}\hat g(\phi,\sigma)\,d\phi$$

If $f$ is essentially $b$-band limited, the essential support of $\hat g_k(\sigma)$ is

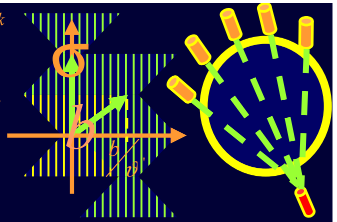
$$\mathbf{K}_2 = \left\{(k,\sigma)\in Z\times R, |\sigma|<b, |k|<\max\left(\frac{|\sigma|}{\upsilon}, b\left(\frac{|1|}{\upsilon}-1\right)\right)\right\}$$




$$\sum_k \int_{(k,\sigma)\notin \mathbf{K}_2} |\hat g_k(\sigma)|\,d\sigma \le \frac{8}{\vartheta\sqrt{2\pi}}\varepsilon_0(f,b) + \eta(\vartheta,b)\|f\|_{L^1}$$

$$0<\vartheta<1$$

This term drives the interpolation error



Let $g\in C_0^\infty\left([0,2\pi)\times\mathbf{R}^{n-1}\right)$ be periodic in its first variable

$$\hat g_k(\sigma)=\frac{1}{\sqrt{2\pi}^n}\int_0^{2\pi}\int_{\mathbf{R}^{n-1}} g(\phi,s)e^{-i(k\phi+s\bullet\sigma)}dsd\phi$$

$$g(\phi,s)=\tilde{\hat g}(\phi,s)=\frac{1}{\sqrt{2\pi}^n}\sum_{-\infty}^{+\infty}\int_{\mathbf{R}^{n-1}}\hat g_k(\sigma)e^{i(k\phi+s\bullet\sigma)}d\sigma$$

The lattice $L_W=\left\{Wl, l\in Z^n\right\}\cap[0,2\pi)\times R^{n-1}$

must be a sub-group of $[0,2\pi)\times\mathbf{R}^{n-1}$ (see Faridani 94
Faridani and Ritman 00)

If $\mathbf{K}+2\pi W^{-t}\mathbf{Z}^n$ are disjoints sets

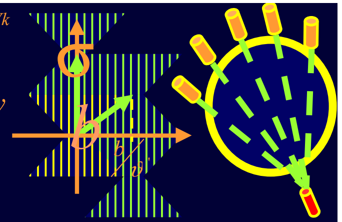
$$S_W g(\varphi,s)=\frac{1}{\sqrt{2\pi}^n}\left|\det W\right|\sum_{y\in L_W} f(y)\tilde\chi_{\mathbf{K}}\left((\varphi,s)-y\right)$$

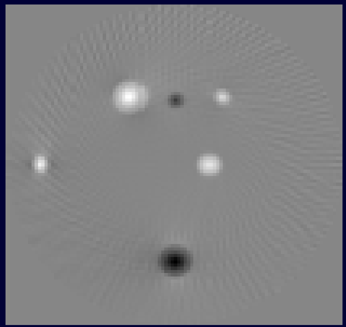
$$\left\|S_W g-g\right\|_\infty\leq 2(2\pi)^{-n/2}\sum\int_{\mathbf{Z}\times\mathbf{R}^{n-1}\setminus\mathbf{K}}\left|\hat g(\sigma)\right|d\sigma$$

# Rattey et Lindgreen 81, Natterer 86

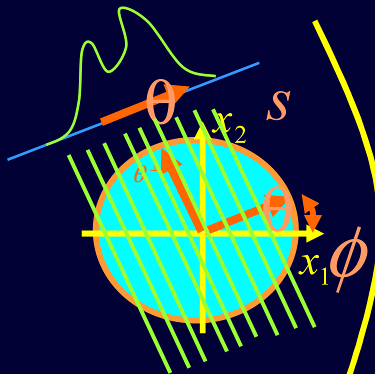
$$\sum_{l\in \mathbf{Z}^n} \hat f\left(\xi - 2\pi W^{-t} l\right) = \frac{1}{\sqrt{2\pi}^n} \left|\det W\right| \sum_{k\in \mathbf{Z}^n} f(Wk) e^{-i\xi . Wk}$$

$$Rf(\theta,s) = \int_{y\in\theta^{\perp}} f(s\theta + y)\,dy$$

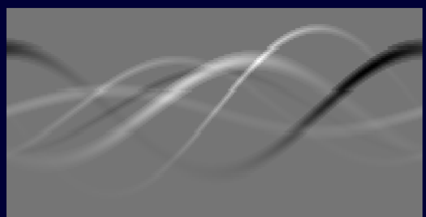


$f(x)$

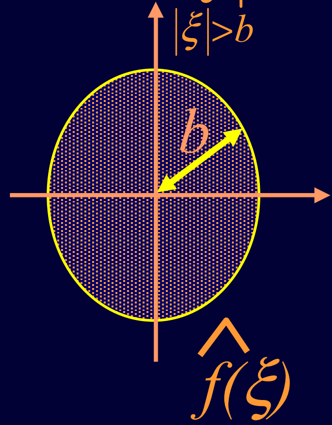


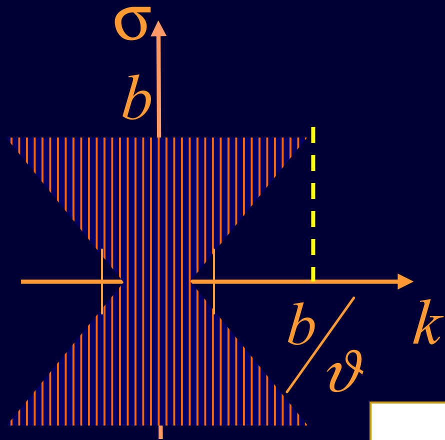

$Rf\,(\phi,s)$

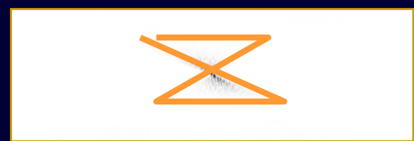


Assumption:

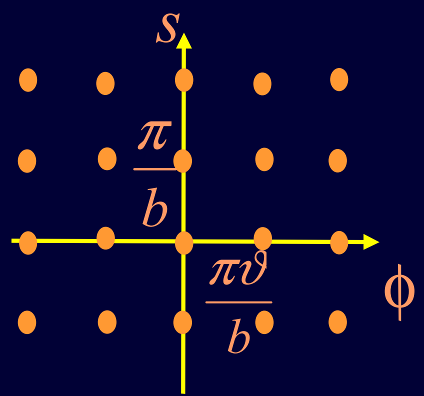
$$\varepsilon_0(f,b) = \int_{|\xi|>b} \left|\hat f(\xi)\right| d\xi < \varepsilon$$



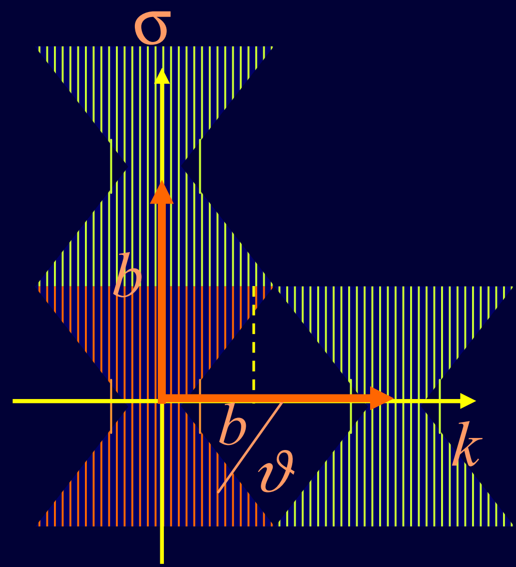



- Standard scheme




Standard




$$W_S=\frac{\pi}{b}\begin{pmatrix}\vartheta & 0\\ 0 & 1\end{pmatrix}$$

$$2\pi W_S^{-t}=2\begin{pmatrix}b/\vartheta & 0\\ 0 & b\end{pmatrix}$$

78 Cormack => interlaced sampling

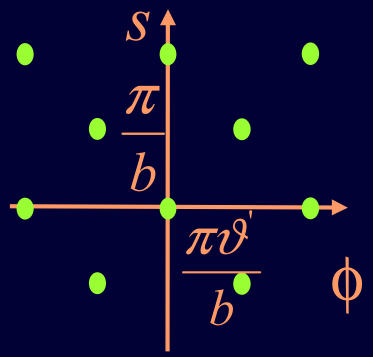
81 Rattey et Lindgren => interlaced sampling and Shannon

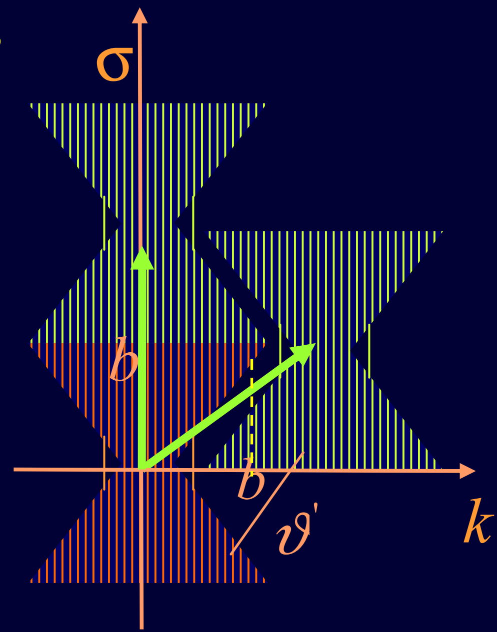
86 Natterer => math. Approach

90,94,00 Faridani => Union of lattices + local tomography

93 Natterer : fan beam sampling conditions

- Interlaced sampling



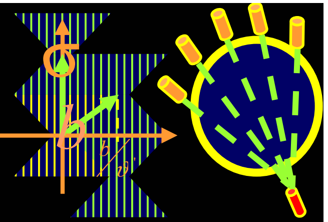

Interlaced

$$W_I = \frac{\pi}{b}\begin{pmatrix} 2\vartheta' & -\vartheta' \\ 0 & 1 \end{pmatrix}$$

$$2\pi W_I^{-t} = \begin{pmatrix} b/\vartheta' & 0 \\ b & 2b \end{pmatrix}$$

$$\sum_{l\in\mathbf{Z}^n}\hat f(\xi-2\pi W^{-t}l)=\tfrac{1}{\sqrt{2\pi}^n}|\det W|\sum_{k\in\mathbf{Z}^n}f(Wk)e^{-i\xi.Wk}$$

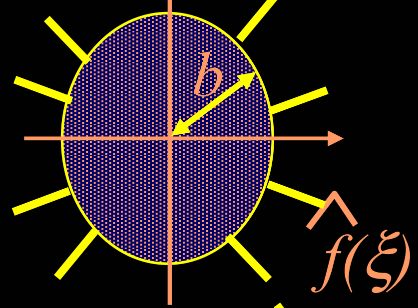
$$Rf(\theta,s)=\int_{y\in\theta^\perp}f(s\theta+y)dy$$



# Sketch of the proof

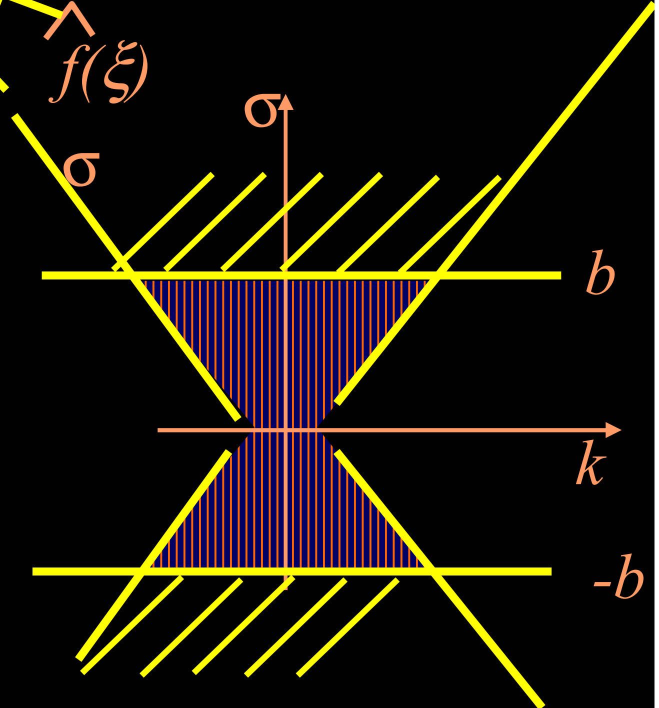
$$\hat g_k(\sigma)=\frac{1}{2\pi}\int_0^{2\pi}e^{-ik\phi}\hat g(\phi,\sigma)d\phi \qquad \hat g(\varphi,\sigma)=\sqrt{2\pi}\,\hat f(\sigma\theta),\ \sigma\in\mathbf{R}$$

$$\varepsilon_0(f,b)=\int_{|\xi|>b}\left|\hat f(\xi)\right|d\xi<\varepsilon$$



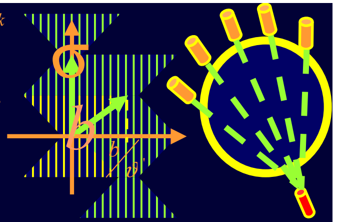

$$\hat g_k(\sigma)=\frac{1}{\sqrt{2\pi}}\int_0^{2\pi}e^{-ik\phi}\hat f(\sigma\theta)d\phi$$

⋮




$$\sum_{|k|<b/\vartheta}\int_{|\sigma|>b}\left|\hat g_k(\sigma)\right|d\sigma\le C_1\varepsilon_0(f,b)$$

$$\sum_{|k|>b/\vartheta}\int_{|\sigma|>\vartheta k}\left|\hat g_k(\sigma)\right|d\sigma\le C_2\varepsilon_0(f,b)$$



$$\hat g_k(\sigma)=\frac{1}{\sqrt{2\pi}}\int_0^{2\pi} e^{-ik\phi}\hat f(\sigma\theta)d\phi$$

⋮

$$\hat g_k(\sigma)=\frac{1}{\sqrt{2\pi}}\, i^k \int_{\Omega_2} f(x)e^{-ik\psi}J_k(-\sigma|x|)dx$$

Bessel function

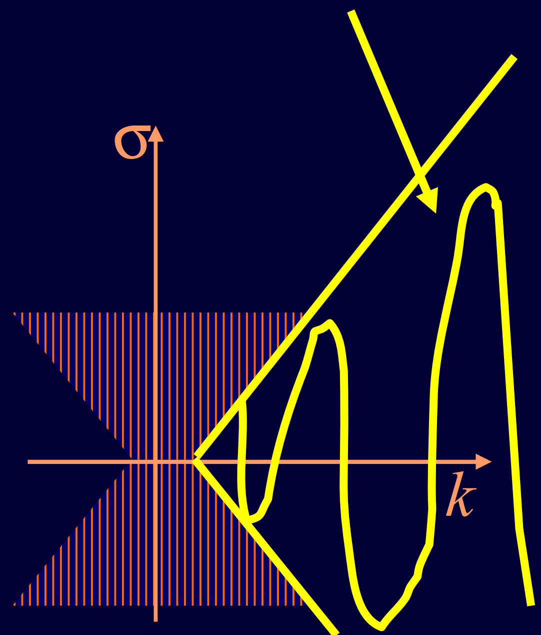
Debye formula

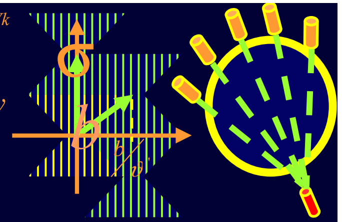
Exponentially decreasing for $|\sigma|\le\vartheta|k|$

$$\int_{|\sigma|<\vartheta|k|} |\hat g_k(\sigma)|d\sigma \le \frac{1}{\sqrt{2\pi}}\eta(\vartheta,|k|)\|f\|_{\mathrm{L}^1}$$

$0<\vartheta<1$

$$0\le\eta(\vartheta,b)\le C(\vartheta)e^{-A(\vartheta)b}$$





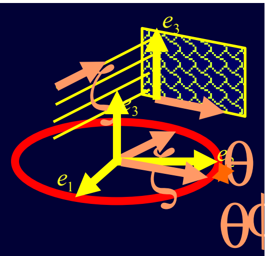
# Generalization

- Rotational invariant RT with polynomial weight

$$g(\phi,s) = Rf(\phi,s) = \int_{-\infty}^{+\infty} f(s\theta + t\theta^\perp) w(s,t) dt$$

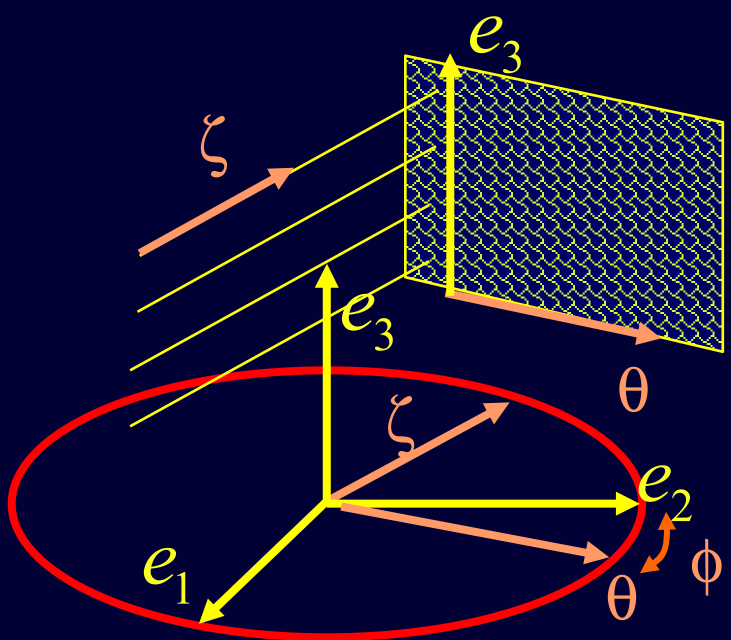
- Exponential Radon transform

$$g(\phi,s) = T_{-\mu} f(\phi,s) = \int_{-\infty}^{+\infty} f(s\theta + t\theta^\perp) e^{-\mu t} dt$$




# Generalization

- 3D X-ray transform (parallel beam) :

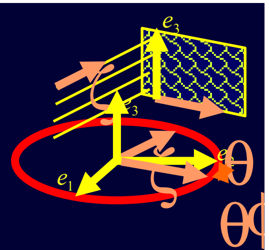


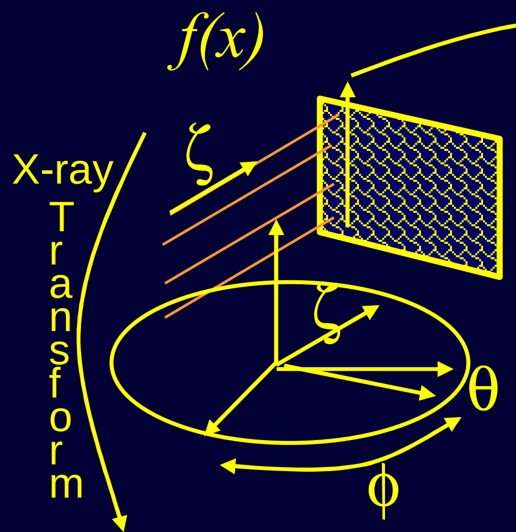

$$Pf(\zeta, x) = \int_{-\infty}^{\infty} f(x + t\zeta)\,dt$$

$$\zeta \in \mathbf{S}^2 \Rightarrow \zeta \in \mathbf{S}^1, x \in \theta^{\perp}$$

$$g(\varphi, s, t) = Pf(\zeta, s\theta + te_3) = \int_{-\infty}^{\infty} f(s\theta + te_3 + u\zeta)\,du$$

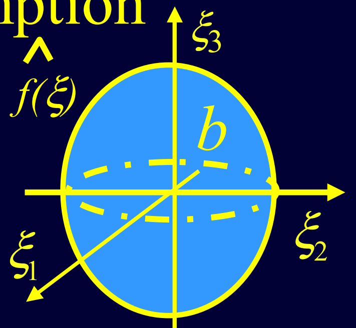
# Sampling conditions






Fourier Transform

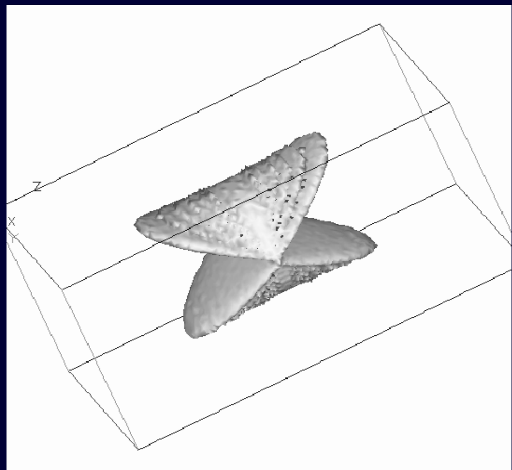
Assumption




$$\varepsilon_0(f,b) = \int_{|\xi|>b} \left|\hat{f}(\xi)\right| d\xi < \varepsilon$$

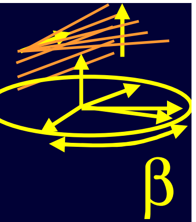
Fourier Transform

$$g(\varphi,s,t) = \int_{-\infty}^{\infty} f(s\theta + te_3 + u\zeta)\,du$$



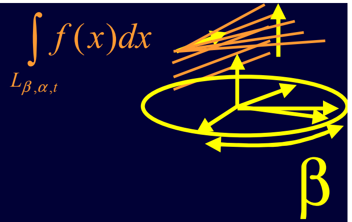
Desbat, 3D95

$\int_{L_{\beta,\alpha,t}} f(x)dx$



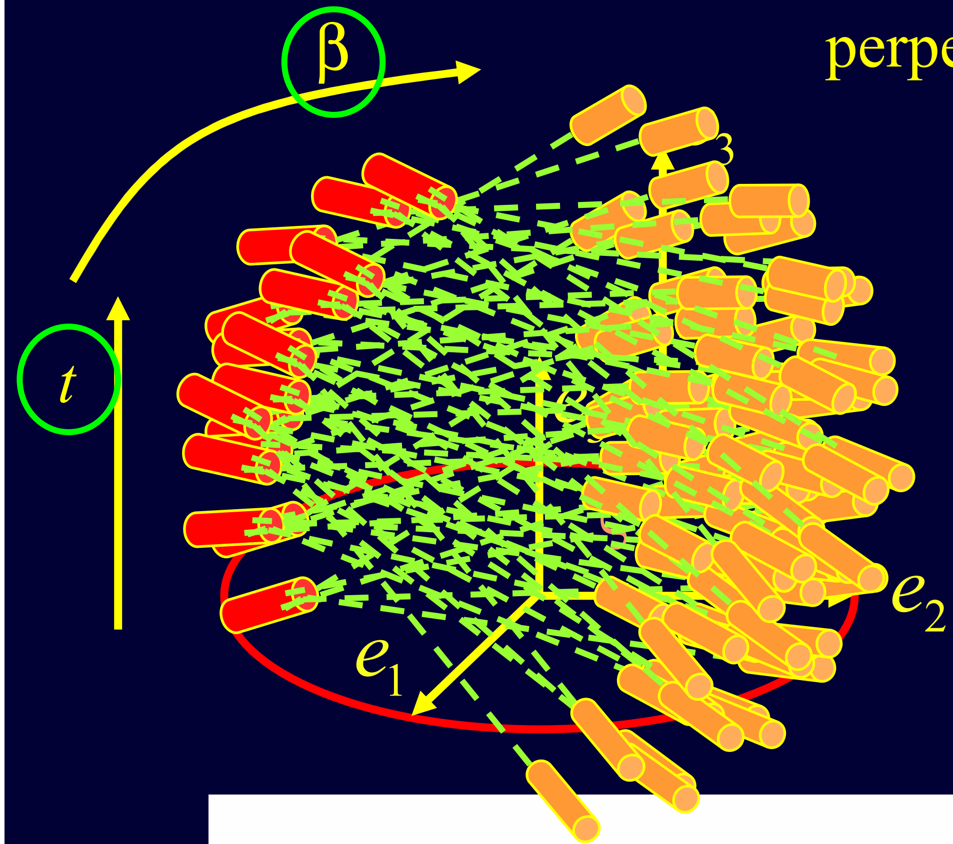

# Plan

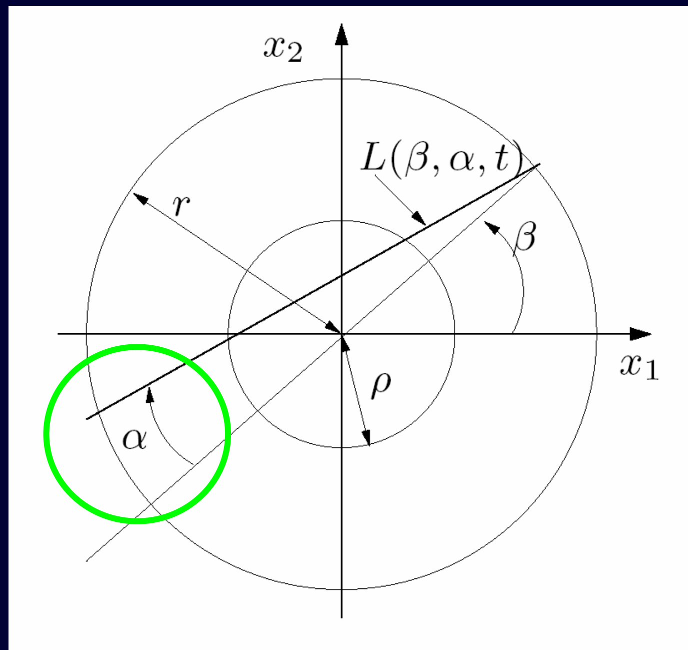
- Tomography
- Sampling
- Sampling in tomography (Shannon approach)
- Sampling the Fan Beam X-ray Transform
- Efficient sampling in parallel helical tomography
- Discussion and perspective




- 3D Fan Beam X-Ray Transform

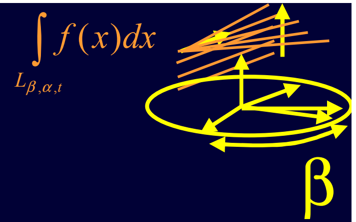
All lines $L_{\alpha,\beta,t}$ are perpendicular to $e_3$




$$g(\beta,\alpha,t) = \mathcal{D}_{e_3{}^\perp} f(\beta,\alpha,t) = \int_{L_{\beta,\alpha,t}} f(x)dx,$$

# Notations

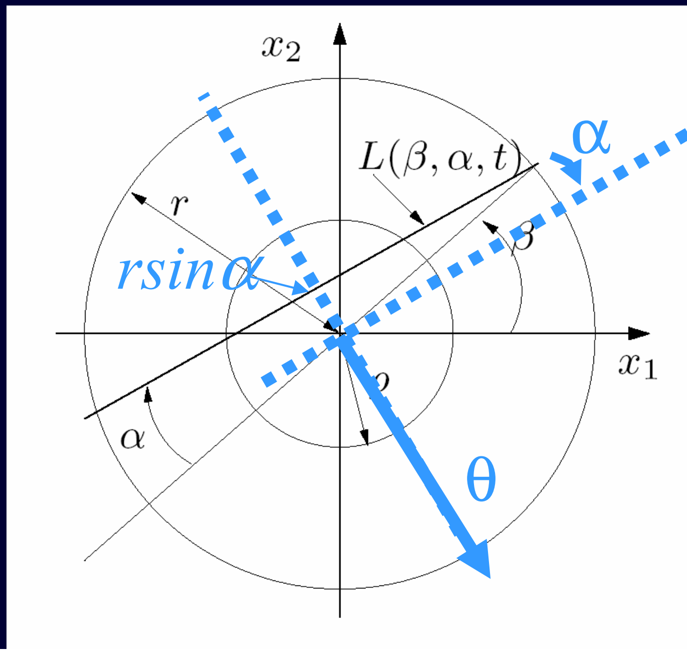


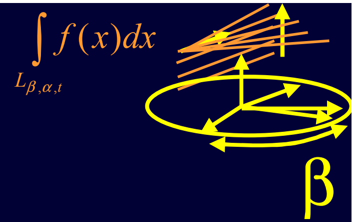

3D Fan Beam X-Ray Transform perpendicular to $e_3$ :

$$g(\beta,\alpha,t) = \mathcal{D}_{e_3{}^\perp} f(\beta,\alpha,t) = \int_{L_{\beta,\alpha,t}} f(x)dx,$$

Link with the parallel 3D X-ray Transform :

$$\mathcal{D}_{e_3{}^\perp} f(\beta,\alpha,t) = \mathcal{P}f(\beta+\alpha-\pi/2, r\sin\alpha, t)$$






# Fourier transform

$$g \in C_0^\infty ([0;2\pi[\times[0;\pi[\times\mathbb{R})$$

$$\hat{g}(\xi) = \frac{1}{2\pi^2\sqrt{2\pi}} \int_{[02\pi[} \int_{[0\pi[} \int_{\mathbb{R}} g(z) e^{-iz\cdot\xi} dz, \quad z\cdot\xi = \beta k + \alpha m + t\tau$$

$$z = (\beta, \alpha, t) \in [0;2\pi[\times[0;\pi[\times\mathbb{R}, \; \xi = (k, m, \tau) \in \mathbb{Z}\times 2\mathbb{Z}\times\mathbb{R}$$

## Inverse Fourier transform

$$\begin{aligned} \check{G}(z) &= (2\pi)^{-1/2} \int_{\mathbb{Z}\times 2\mathbb{Z}\times\mathbb{R}} G(\xi) e^{iz\cdot\xi} \\ &= (2\pi)^{-1/2} \sum_{k\in\mathbb{Z}} \sum_{m\in 2\mathbb{Z}} \int_{\tau\in\mathbb{R}} G(k,m,\tau) e^{i(k\beta + m\alpha + \tau t)} d\sigma. \end{aligned}$$

# Fourier interpolation

Let $\mathbf{K} \subset \mathbb{Z} \times 2\mathbb{Z} \times \mathbb{R}$, the non-overlapping Shannon condition associated to $\mathbf{K}$ for the sampling lattice $L_W = W\mathbb{Z}^3 \cap ([0; 2\pi[\times[0; \pi[\times\mathbb{R})$ generated by the non singular $3 \times 3$ matrix $W$ is that *the sets* $\mathbf{K} + 2\pi W^{-t} l, l \in \mathbb{Z}^3$ *are disjoint sets in* $\mathbb{Z} \times 2\mathbb{Z} \times \mathbb{R}$. The Petersen-Middleton theorem [10, 5] yields the Fourier interpolation formula

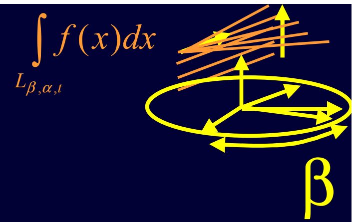
$$(S_W g)(z) = (2\pi)^{-1/2} |\det W| \sum_{y \in L_W} g(y) \check{\chi}_{\mathbf{K}}(z - y) \tag{4}$$

with the interpolation error

$$||S_W g - g||_\infty \leq 2(2\pi)^{-1/2} \int_{\xi \notin \mathbf{K}} |\hat{g}(\xi)| d\xi.$$

Shannon : $\mathbf{K} + 2\pi W^{-t}\mathbf{Z}^n$ 2 à 2 disjoints.

Support essentiel de $|\hat{g}(\xi)|$

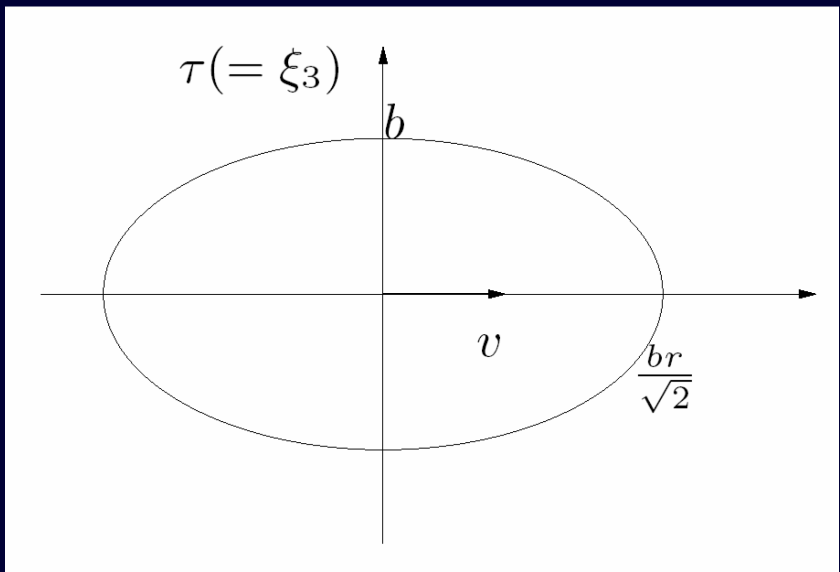


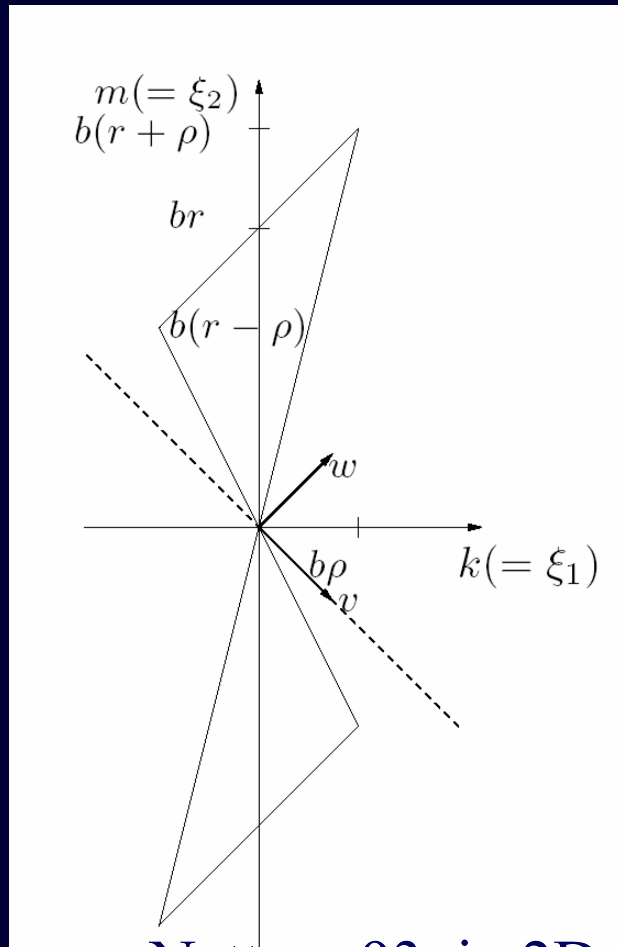

# Main result

$$K_{\mathcal{D}_{e_3{}^\perp}} = \{(k,m,\tau) \in \mathbb{Z} \times 2\mathbb{Z} \times \mathbb{R};\, |k-m|^2 + r^2\tau^2 < r^2 b^2,\, |k|r < |k-m|\rho\}$$

$|\hat{g}(\xi)|$ negligible outside of $K_{D_{e_3^\perp}}$

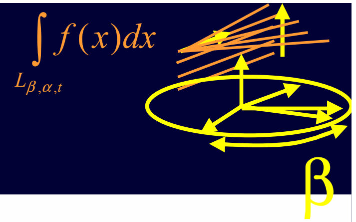
if $f$ is essentially $b$ band-limited




Natterer93, in 2D

# Sketch of the proof




$$\hat{g}(k,m,\tau) = \frac{1}{4\pi^2\sqrt{2\pi}} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\infty}^{\infty} \mathcal{P}f(\beta+\alpha-\pi/2, r\sin\alpha, t) e^{-i(k\beta+m\alpha+\tau t)} d\beta d\alpha dt$$

$$= \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \widehat{\mathcal{P}f}^{3}(\beta+\alpha-\pi/2, r\sin\alpha, \tau) e^{-i(k\beta+m\alpha)}, d\beta d\alpha$$

$$\widehat{\mathcal{P}f}(\beta+\alpha-\pi/2,\sigma,\tau) = \sqrt{2\pi}\hat{f}(\sigma\theta(\beta+\alpha-\pi/2)+\tau e_3)$$

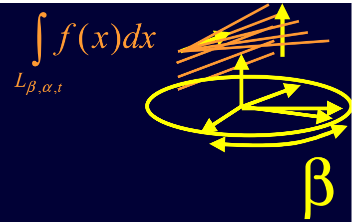
$$\widehat{\mathcal{P}f}^{3}(\beta+\alpha-\pi/2, r\sin\alpha, \tau)$$

$$= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \widehat{\mathcal{P}f}(\beta+\alpha-\pi/2,\sigma,\tau) e^{i\sigma r\sin\alpha} d\sigma$$

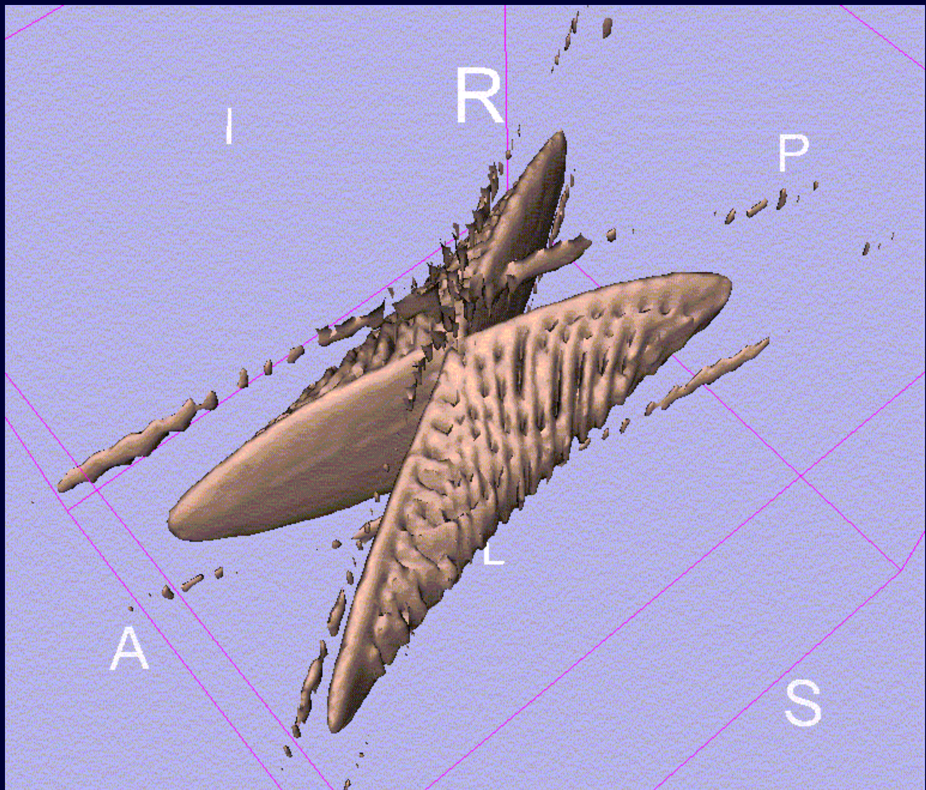
$$= \int_{\mathbb{R}} \hat{f}(\sigma\theta(\beta+\alpha-\pi/2)+\tau e_3) e^{i\sigma r\sin\alpha} d\sigma$$

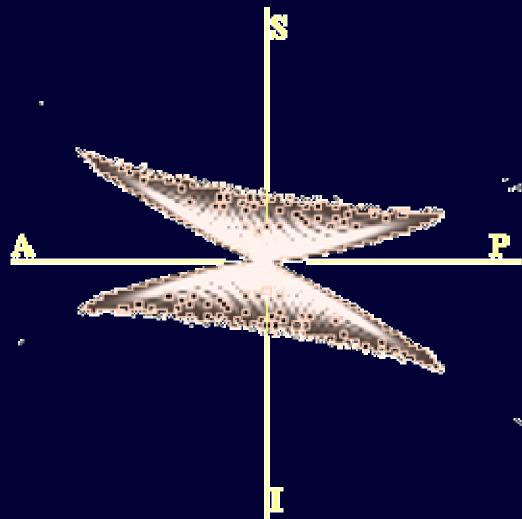
$$= \frac{1}{2\pi}\int_{\mathbb{R}^2} \hat{f}^{3}(x_1,x_2,\tau) \int_{\mathbb{R}} e^{-ix\cdot\theta(\beta+\alpha-\pi/2)+i\sigma r\sin\alpha} d\sigma dx_1 dx_2$$

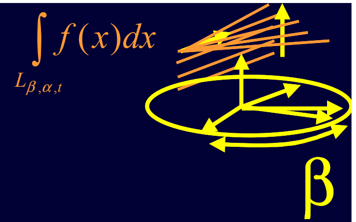
$\leftarrow$ $\sqrt{\sigma^2+\tau^2} < b$



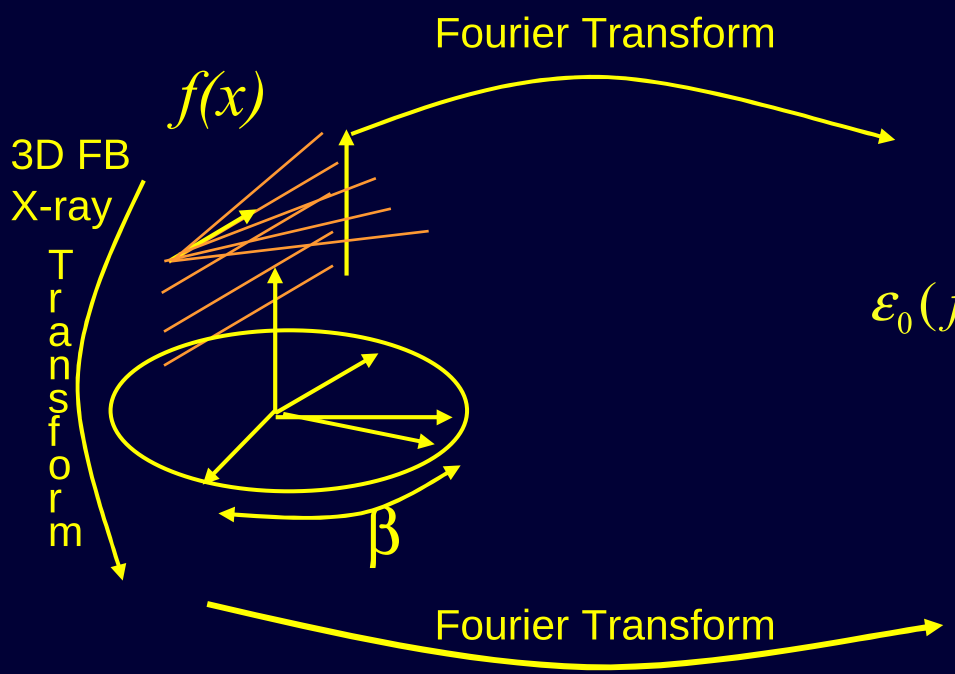

# Essential support of $\left|\hat{g}(\xi)\right|$

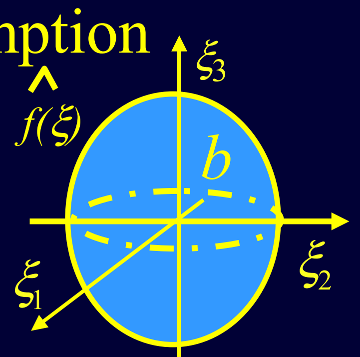


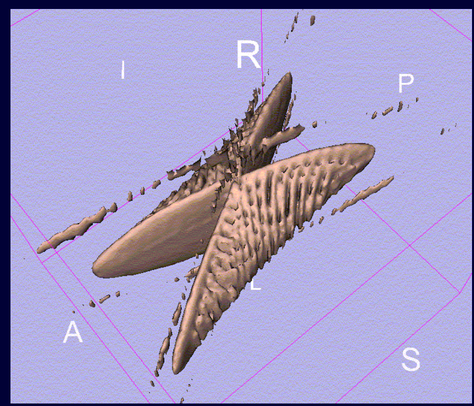



# 3D // Fan Beam sampling conditions



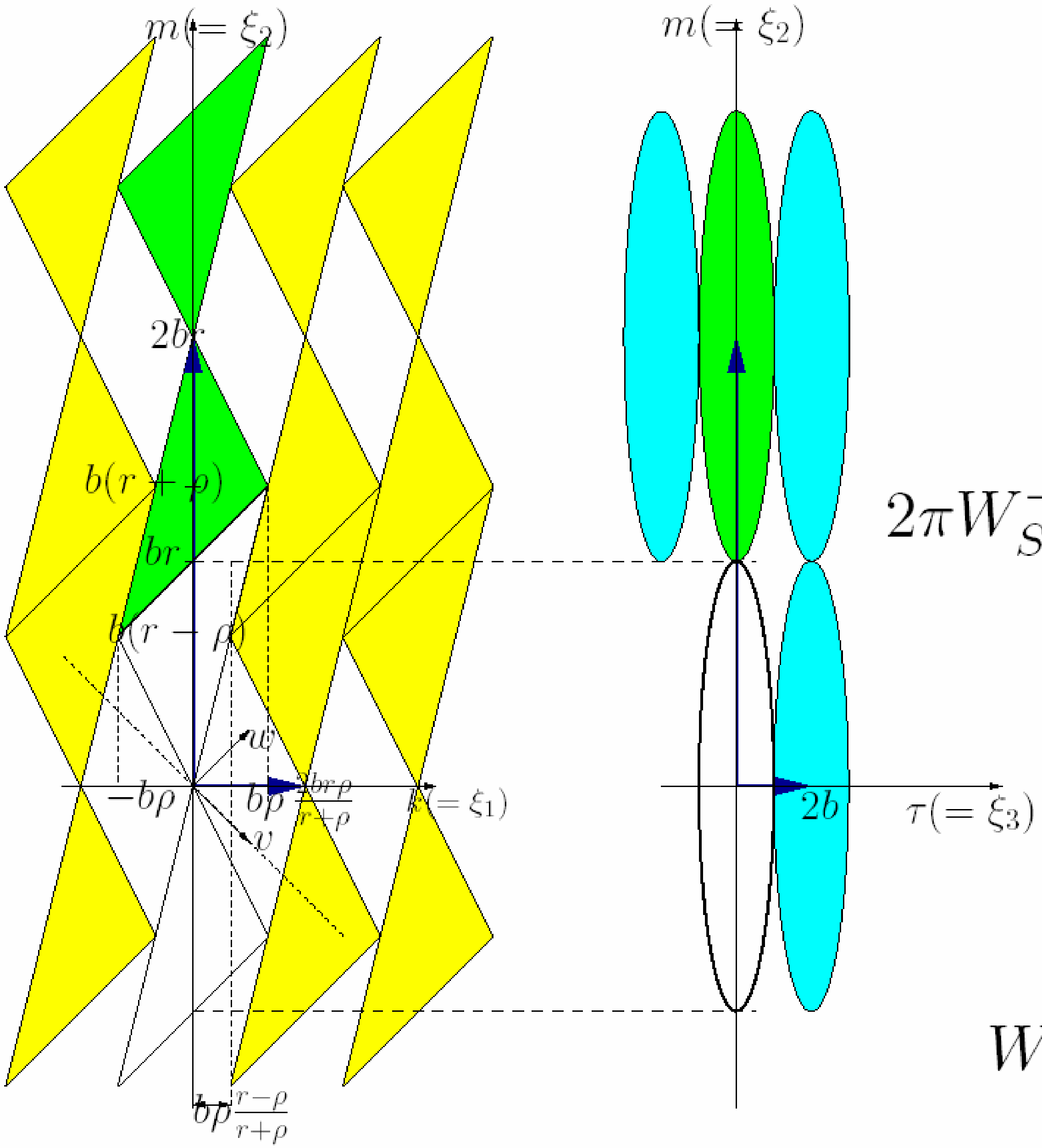




$$\varepsilon_0(f,b) = \int_{|\xi|>b} \left|\hat{f}(\xi)\right| d\xi < \varepsilon$$




$$g(\beta,\alpha,t) = \mathcal{D}_{e_3{}^\perp} f(\beta,\alpha,t) = \int_{L_{\beta,\alpha,t}} f(x)dx,$$

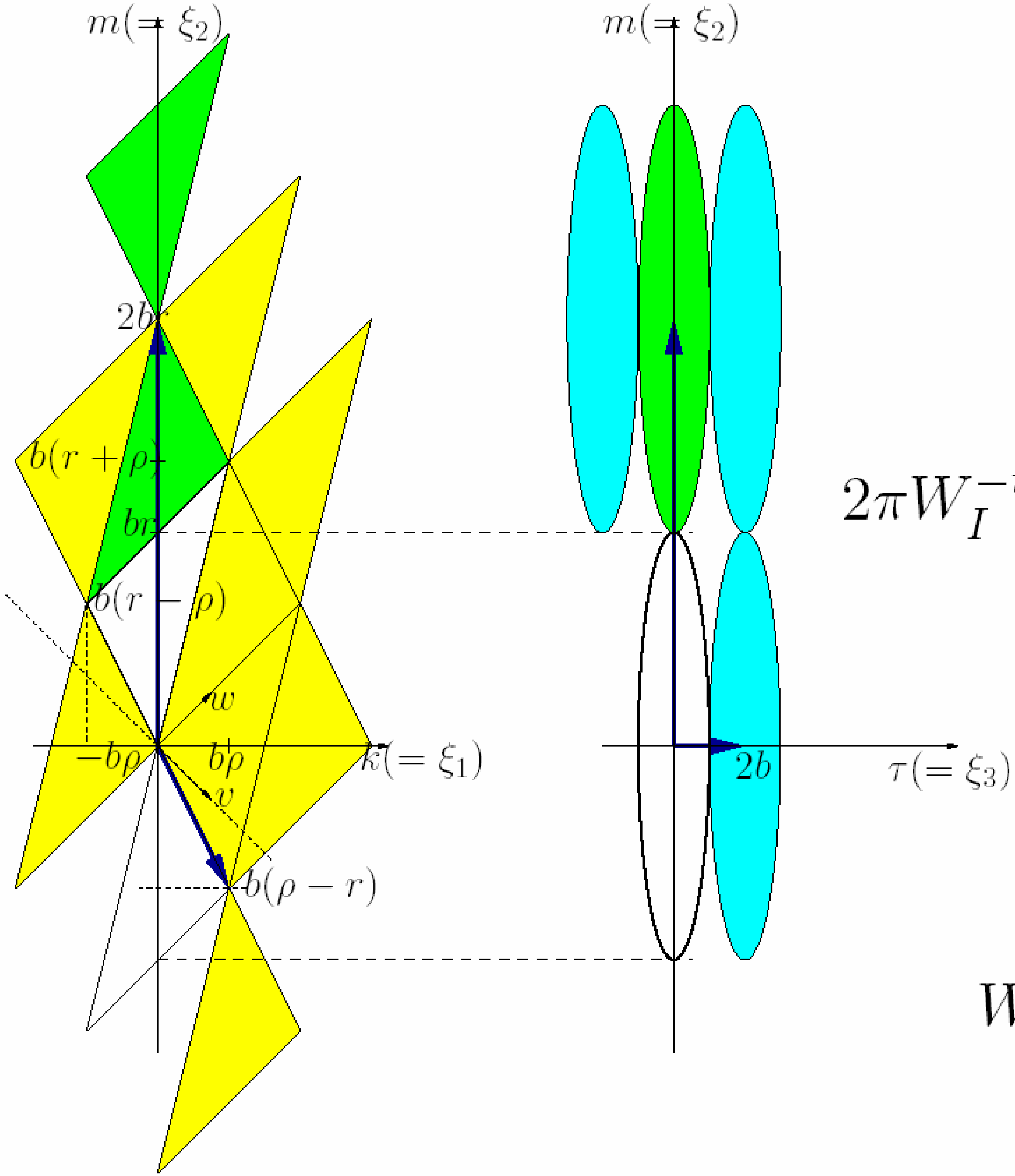
# Standard sampling schemes




$$2\pi W_S^{-t} = 2b \begin{bmatrix} \frac{r\rho}{\rho+r} & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$W_S = \frac{\pi}{b} \begin{bmatrix} \frac{\rho+r}{r\rho} & 0 & 0 \\ 0 & \frac{1}{r} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

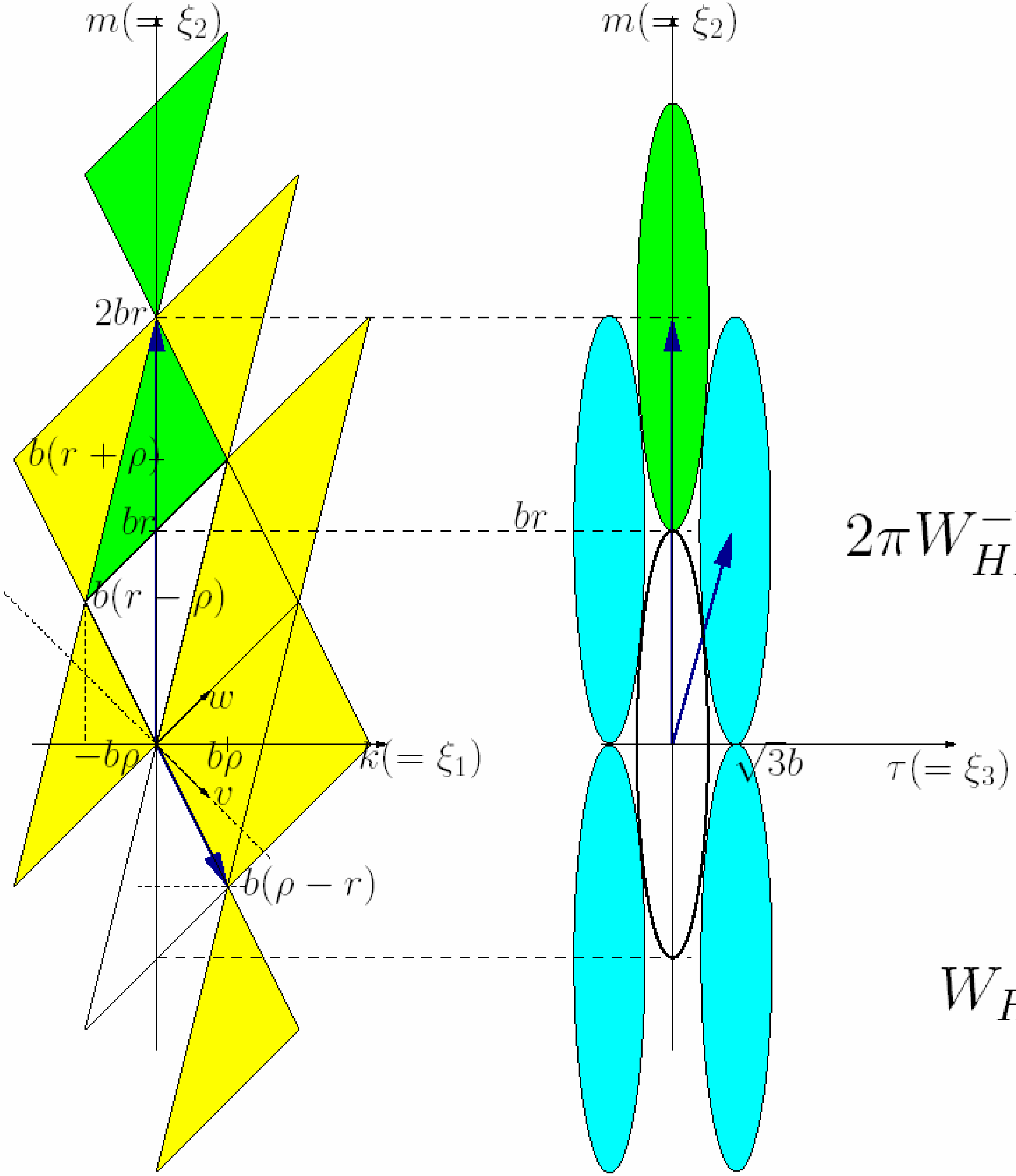
# Interlaced sampling schemes




$$2\pi W_I^{-t} = b \begin{bmatrix} \rho & 0 & 0 \\ \rho - r & 2r & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$W_I = \frac{\pi}{b} \begin{bmatrix} \frac{2}{\rho} & \frac{r-\rho}{r\rho} & 0 \\ 0 & \frac{1}{r} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

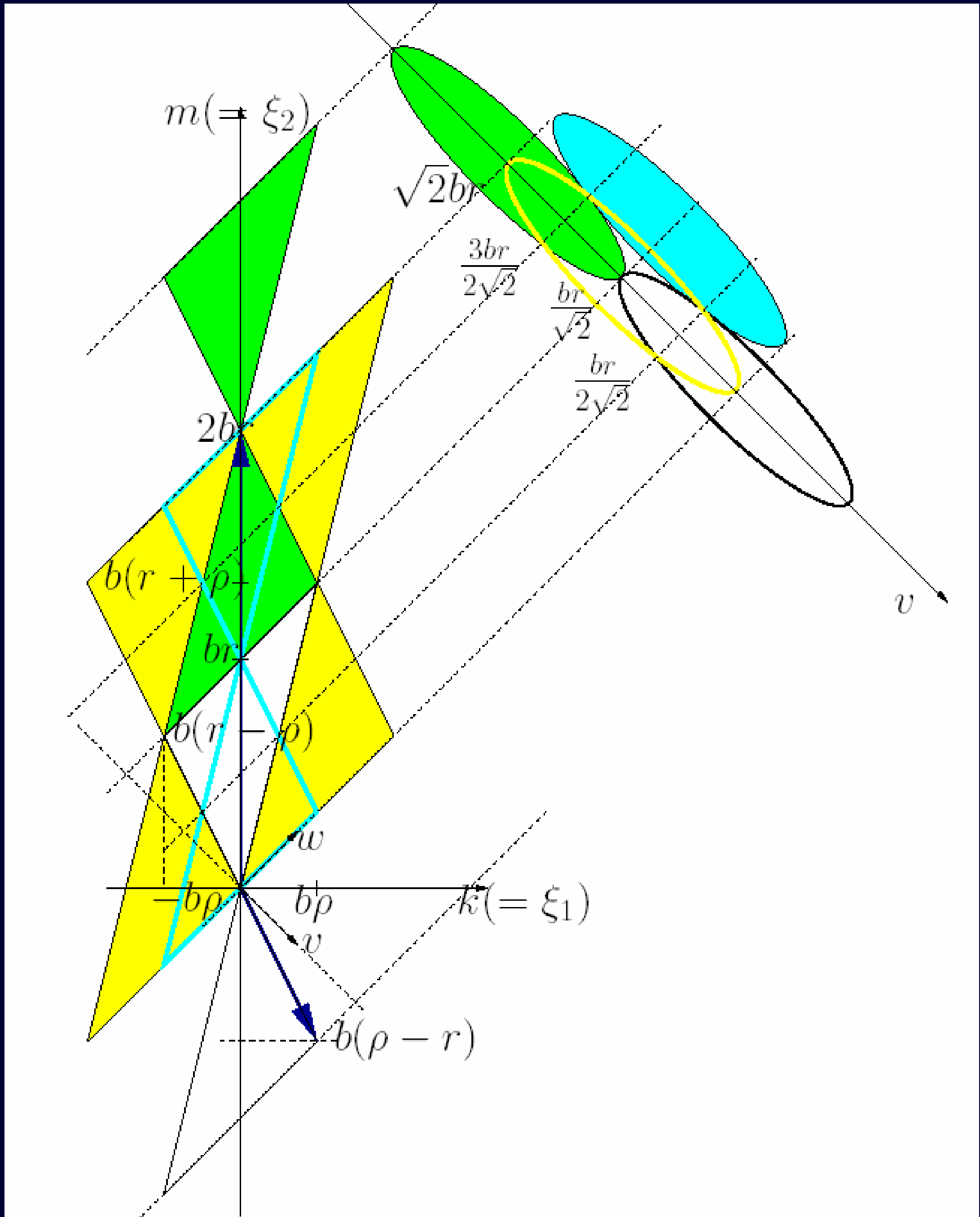
# Hexagonal Interlaced sampling schemes

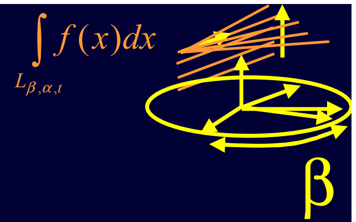



$$2\pi W_{HI}^{-t} = b \begin{bmatrix} \rho & 0 & 0 \\ \rho - r & 2r & r \\ 0 & 0 & \sqrt{3} \end{bmatrix}$$

$$W_{HI} = \frac{\pi}{b} \begin{bmatrix} \frac{2}{\rho} & \frac{r-\rho}{r\rho} & 0 \\ 0 & \frac{1}{r} & 0 \\ 0 & \frac{-1}{\sqrt{3}} & \frac{2}{\sqrt{3}} \end{bmatrix}$$

# HI schemes discussion





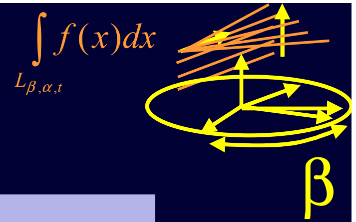

# Schemes efficiency

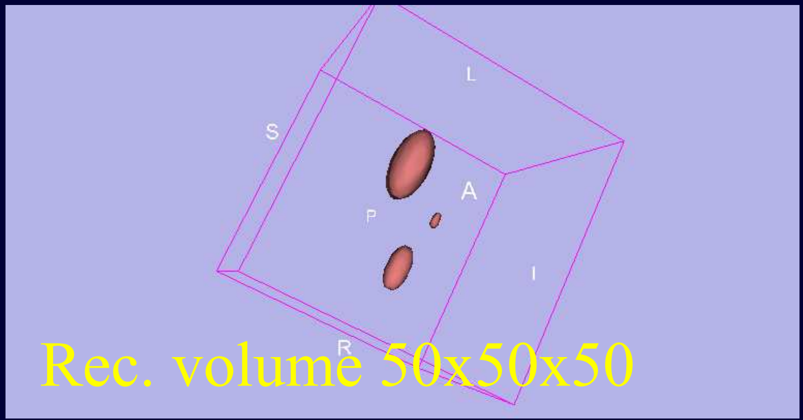
$$|\det W_{HI}| = \frac{2}{\sqrt{3}}|\det W_I| = \frac{2}{\sqrt{3}}\frac{2\eta}{\eta^2+\eta}|\det W_S|$$

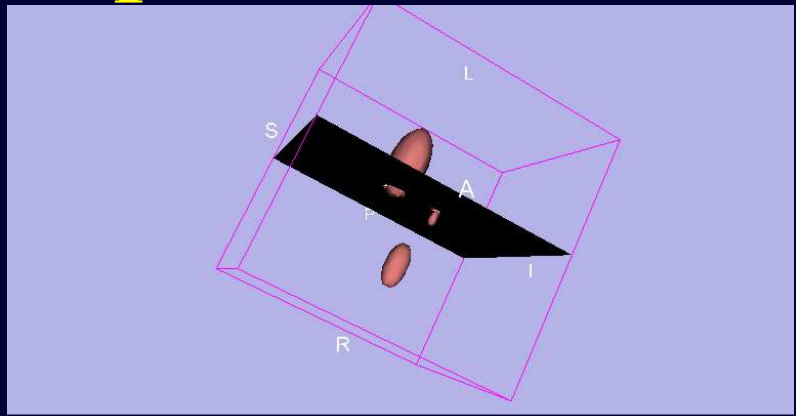
$$\eta = \frac{\rho}{r} \leq 1$$

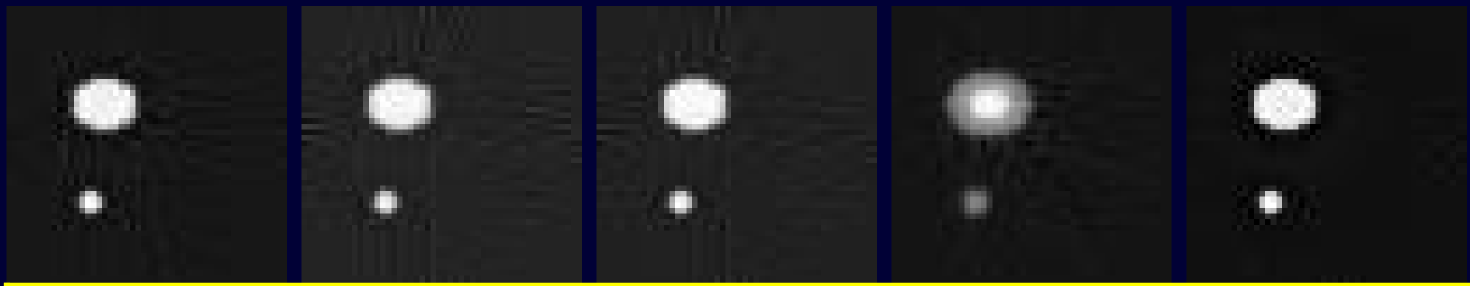
$$\left|\det W_{HI}\right| > \left|\det W_I\right| \geq \left|\det W_S\right|$$

# Numerical experiments

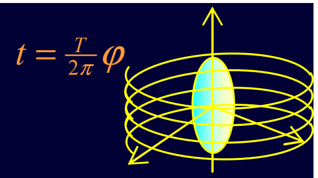


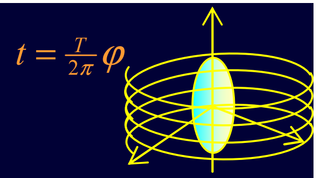
| S | I | HI | S | HI | |
|---|---|---|---|---|---|
| 128 | 345 | 172 | 89 | 272 | $N_\beta$ |
| 51 | 7 | 7 | 36 | 9 | $N_\alpha$ |
| 51 | 51 | 88 | 36 | 128 | $N_t$ |
| 293760 | 102000 | 89000 | 101556 | 274688 | Data |
| 2,27 | 2,92 | 3,03 | 6,07 | 1,88 | $L^2$ error*100 |

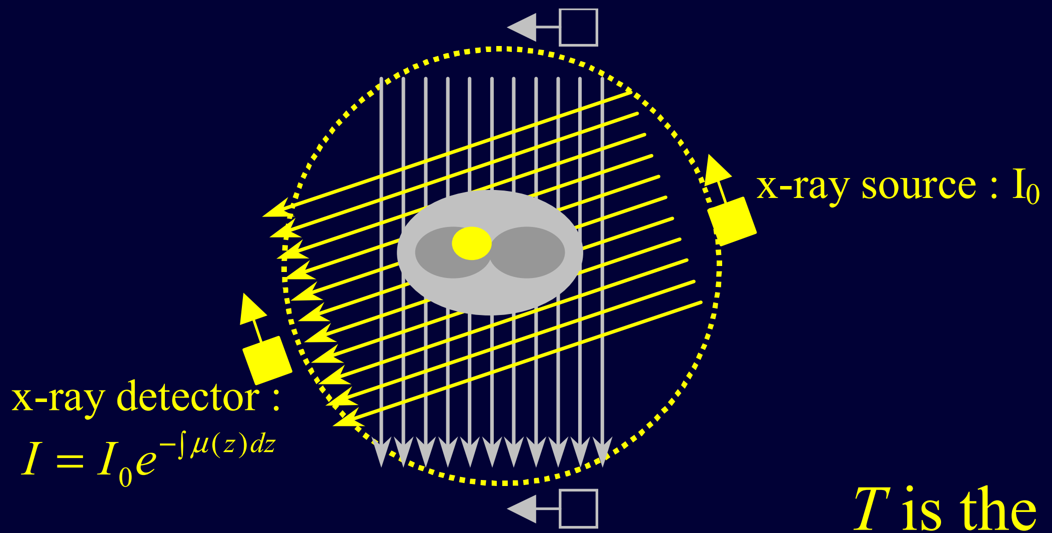


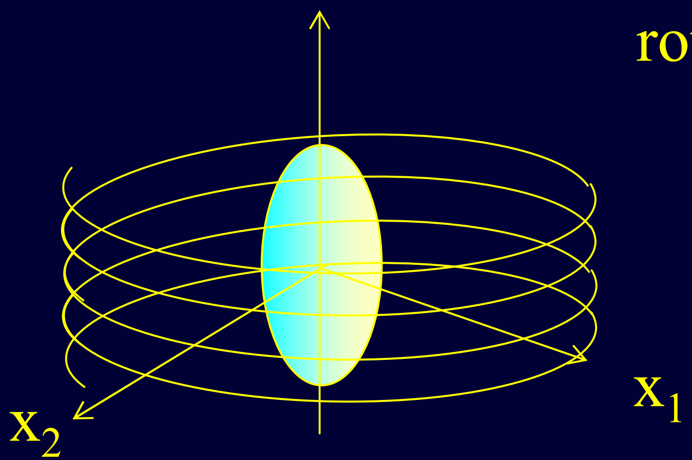

# Plan

- Tomography
- Sampling
- Sampling in tomography (Shannon approach)
- Sampling the Fan Beam X-ray Transform
- Efficient sampling in parallel helical tomography
- Discussion and perspective

# Parallel helical tomography

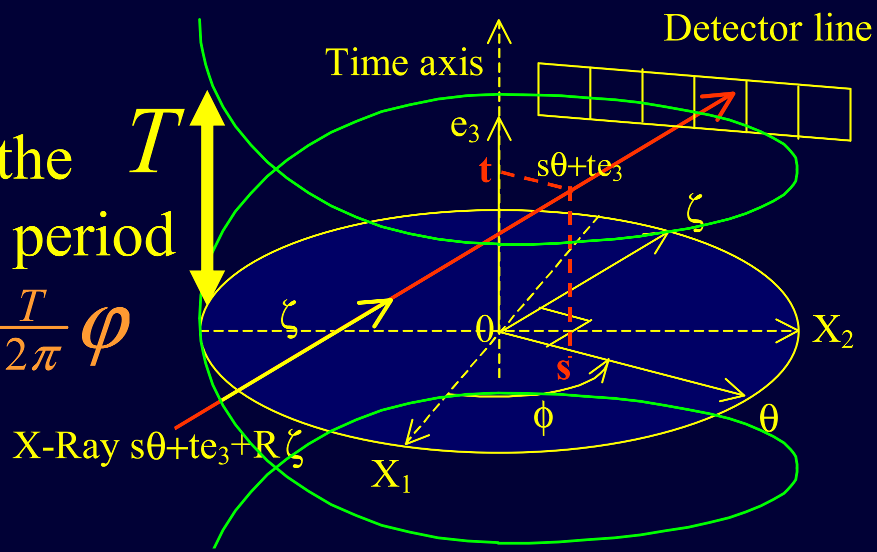


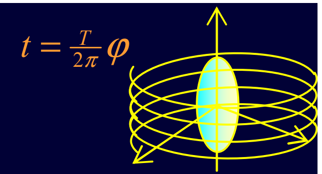




*T* is the rotation period

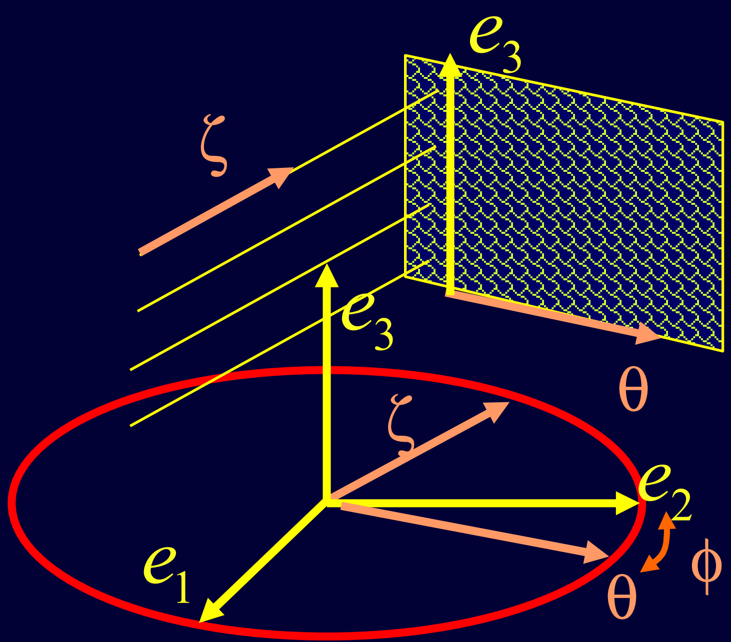
$$t = \frac{T}{2\pi}\varphi$$




$$g(\varphi, s, t) = Pf(\zeta, s\theta + te_3) = \int_{-\infty}^{\infty} f(s\theta + te_3 + u\zeta)\,du$$




- 3D // X Ray Transform ：

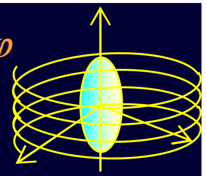



$$Pf(\zeta, x) = \int_{-\infty}^{\infty} f(x + t\zeta)\,dt$$

$$\zeta \in \mathbf{S}^2 \Rightarrow \zeta \in \mathbf{S}^1, x \in \theta^{\perp}$$

$$g(\varphi, s, t) = Pf(\zeta, s\theta + te_3) = \int_{-\infty}^{\infty} f(s\theta + te_3 + u\zeta)\,du$$

$t = \frac{T}{2\pi}\varphi$

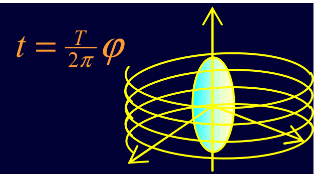


- Fourier transform:

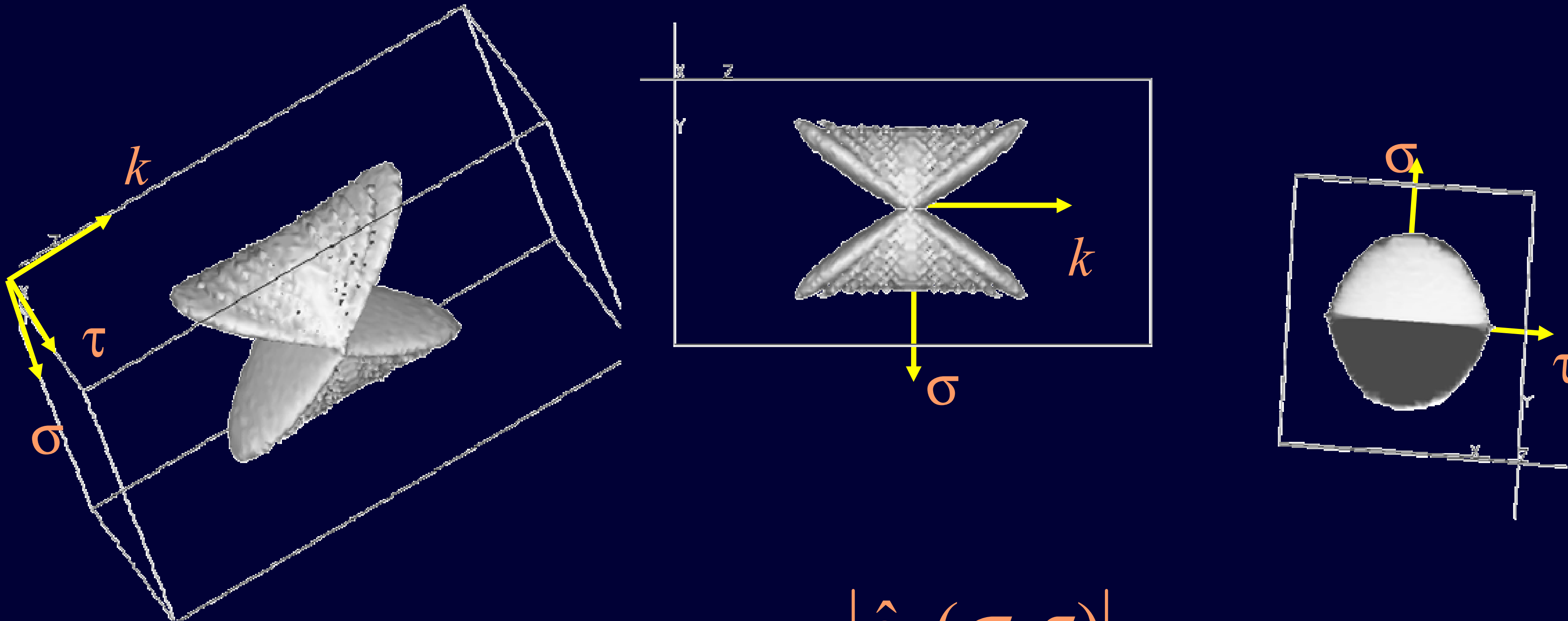
$$g(\varphi, s, t) = \int_{-\infty}^{\infty} f(s\theta + te_3 + u\zeta)\,du$$

$$\hat{g}(\varphi, \sigma, \tau) = \frac{1}{2\pi}\int_{\mathbf{R}^2} g(\varphi, s, t)e^{-i(\sigma s + t\tau)}\,ds\,dt$$

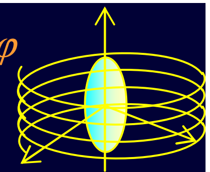
$$\hat{g}_k(\sigma, \tau) = \frac{1}{2\pi}\int_0^{2\pi} \hat{g}(\varphi, \sigma, \tau)e^{-ik\varphi}\,d\varphi, k \in \mathbf{Z}$$




- Essential support of the Fourier transform of the 3D//XRT:



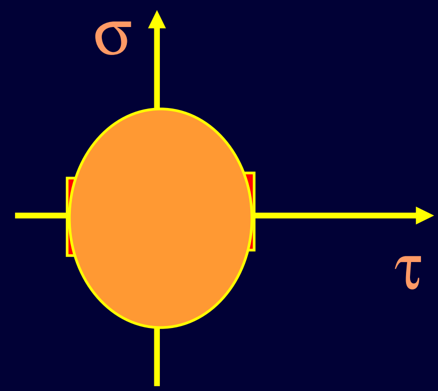

$$\left|\hat{g}_k(\sigma,\tau)\right|$$



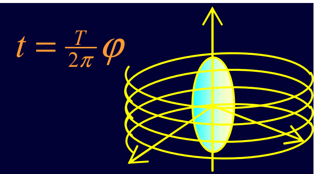

- Essential support of the Fourier transform of the 3D//XRT:

$$\mathbf{K}_3 = \left\{ (k,\sigma,\tau) \in \mathbf{Z}\times\mathbf{R}\times\mathbf{R}, |\sigma| < b, |k| < \max\left(\frac{|\sigma|}{\upsilon}, b\left(\frac{|1|}{\upsilon} - 1\right)\right), \tau < c(b,\sigma) \right\}$$

$$c(b,\sigma) = \begin{cases} b & \text{si} \, |\sigma| < \sigma_{\vartheta,b} \max(1,(1-\upsilon)b) \\ \sqrt{b^2 - \sigma^2} & \text{si} \, \sigma_{\vartheta,b} \le |\sigma| < b \end{cases}$$

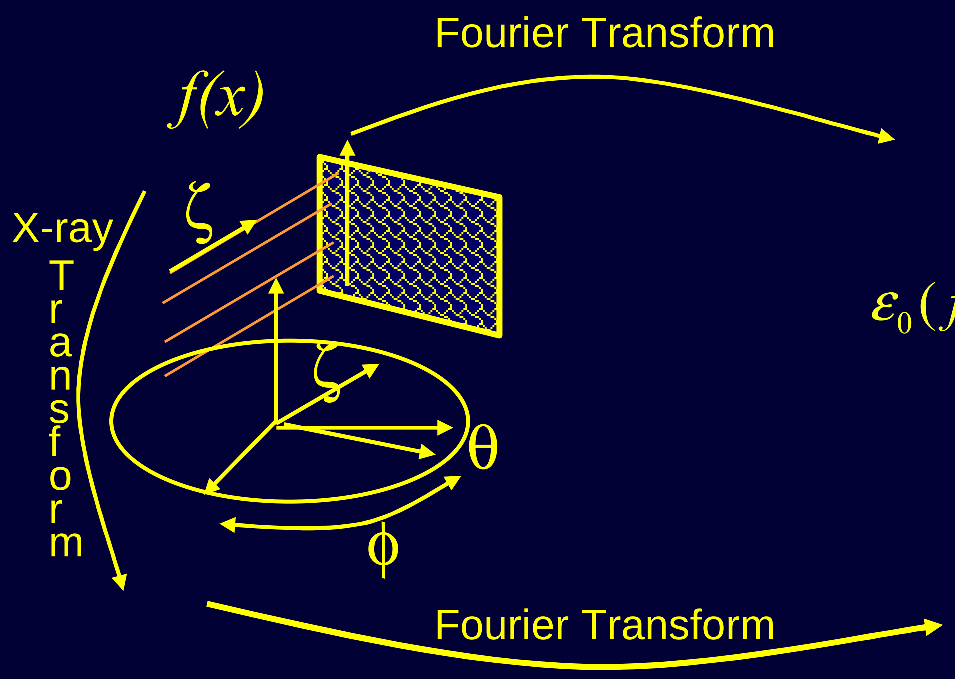



$$\sum_k \int_{(k,\sigma,\tau)\notin \mathbf{K}_3} \left| \hat{g}_k(\sigma,\tau) \right| d\sigma d\tau \le C_1 \eta(\vartheta,(1/\vartheta - 1)b) + C_2 \varepsilon_0(f,b)$$

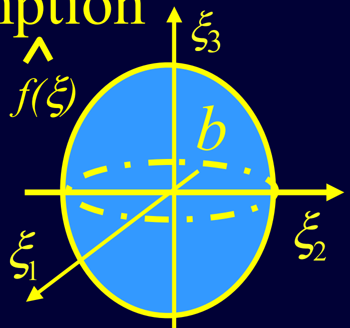



# 3D // sampling conditions

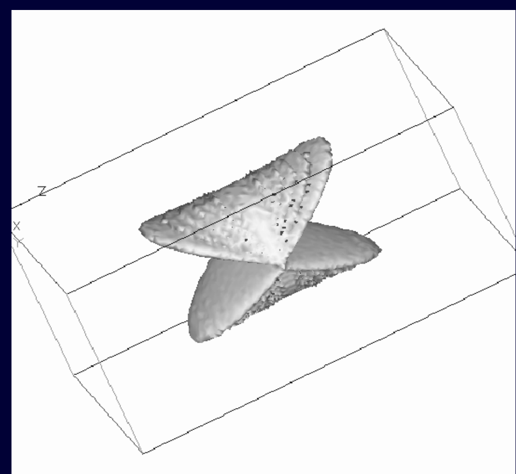
Desbat95



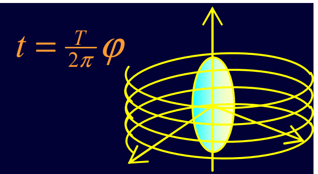

Assumption



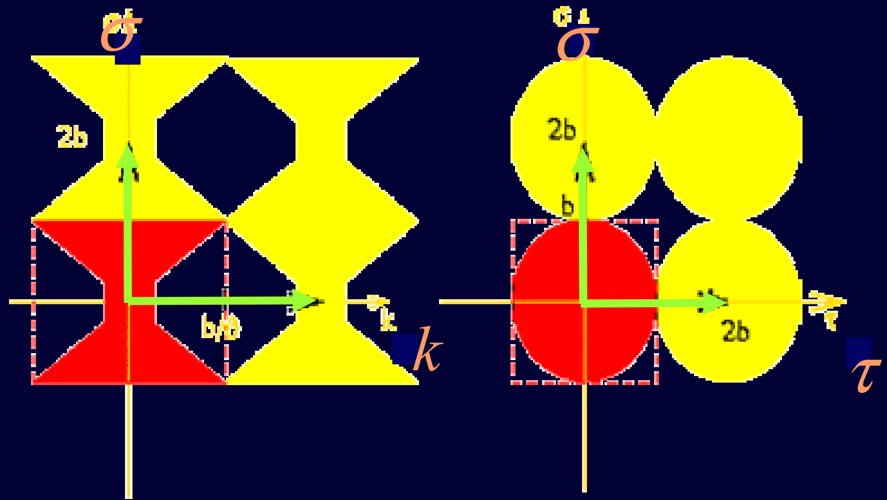

$$\varepsilon_0(f,b) = \int_{|\xi|>b} \left|\hat{f}(\xi)\right| d\xi < \varepsilon$$



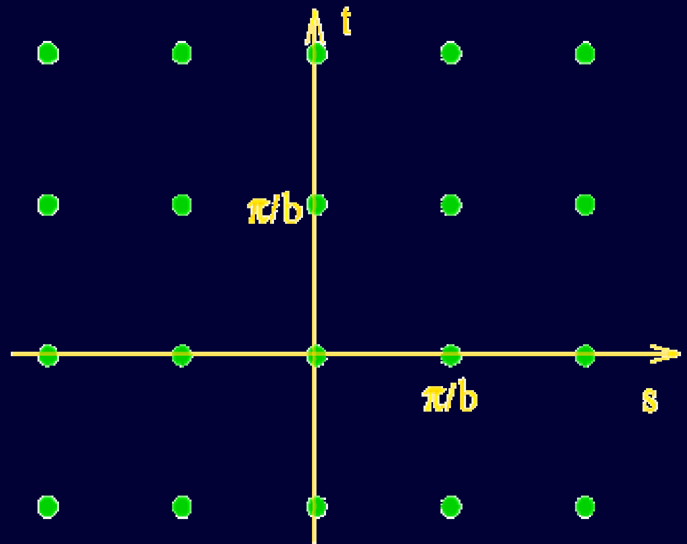
$$g(\varphi,s,t) = \int_{-\infty}^{\infty} f(s\theta + te_3 + u\zeta)\,du$$




- Standard Scheme

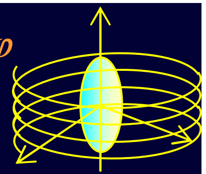



$$2\pi W_S^{-t} = 2b\begin{pmatrix} 1/\vartheta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

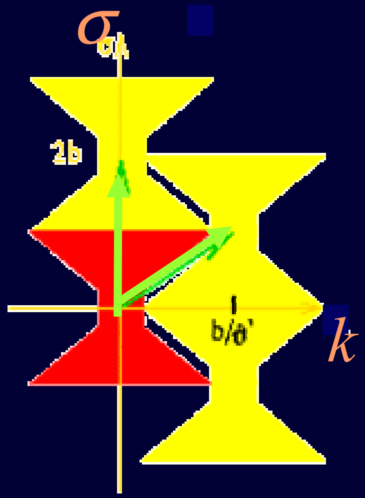


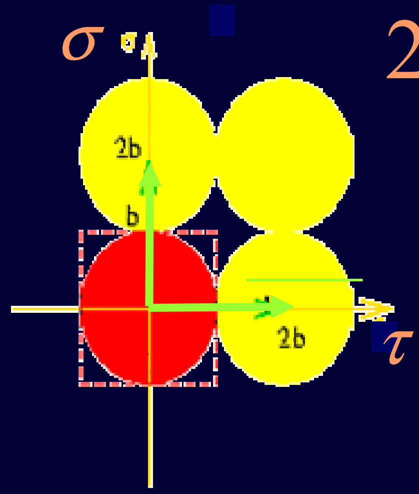

$$W_S = \frac{\pi}{b}\begin{pmatrix} \vartheta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Interlaced scheme



$t = \frac{T}{2\pi}\varphi$

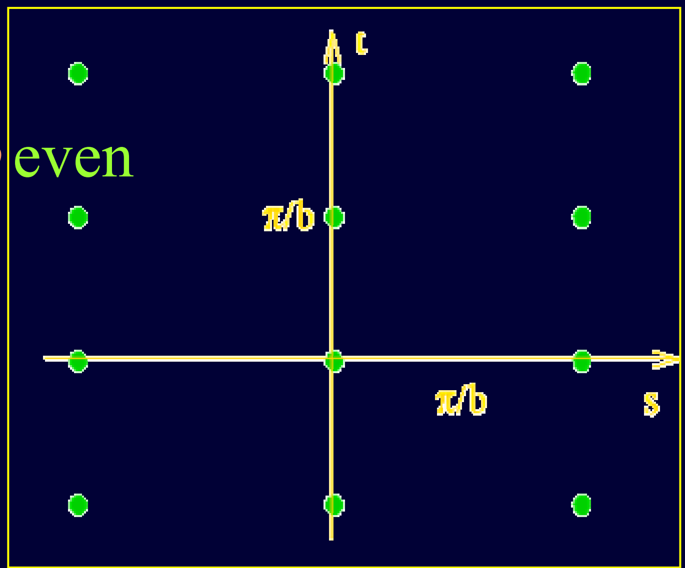


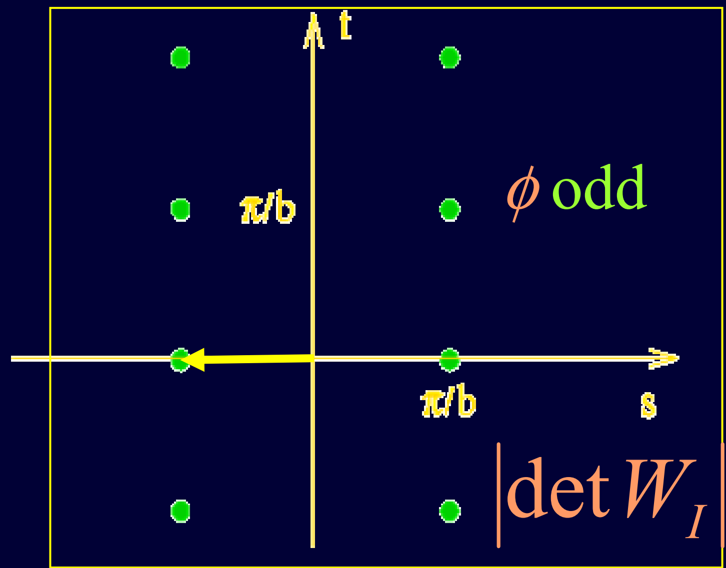




$$2\pi W_I^{-t} = b\begin{pmatrix} 1/\vartheta' & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$W_I = \frac{\pi}{b}\begin{pmatrix} 2\vartheta' & -\vartheta' & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\phi$ even

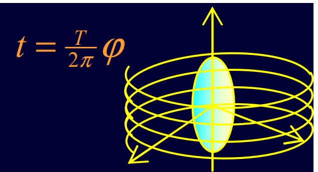


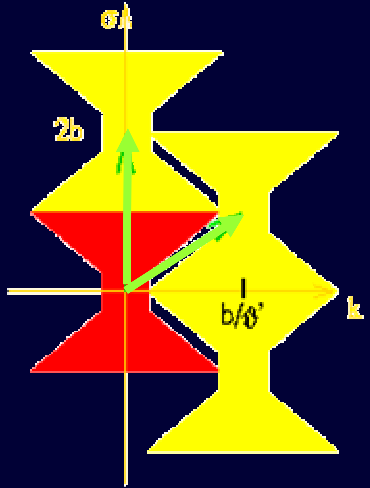



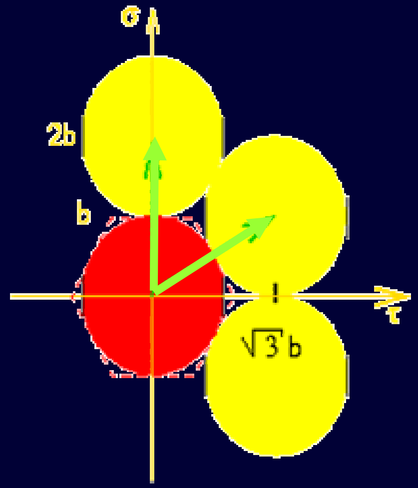

$\phi$ odd

$$|\det W_I| \approx 2|\det W_S|$$

- Hexagonal Interlaced scheme

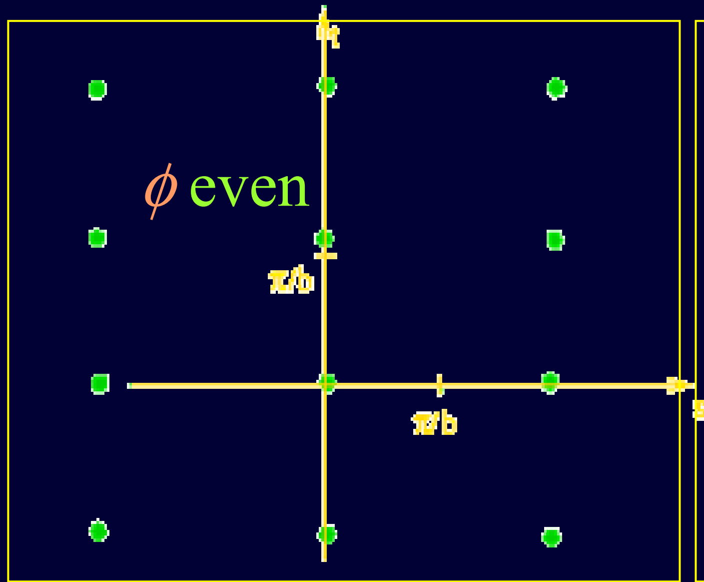


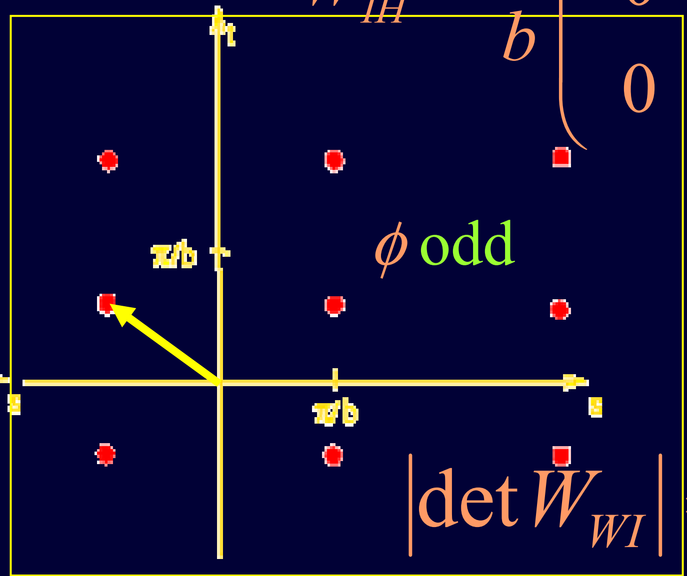




$$2\pi W_{IH}^{-t} = b\begin{pmatrix} 1/\vartheta' & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & \sqrt{3} \end{pmatrix}$$

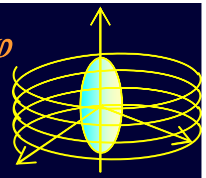
$$W_{IH} = \frac{\pi}{b}\begin{pmatrix} 2\vartheta' & -\vartheta' & 0 \\ 0 & 1 & 0 \\ 0 & \frac{-1}{\sqrt{3}} & \frac{2}{\sqrt{3}} \end{pmatrix}$$



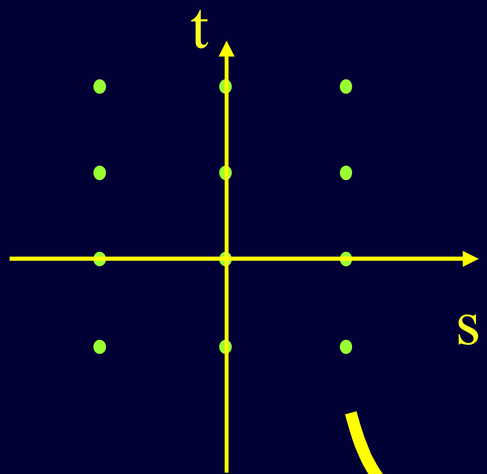




$$|\det W_{WI}| \approx \frac{4}{\sqrt{3}}|\det W_S|$$

$t = \frac{T}{2\pi}\varphi$

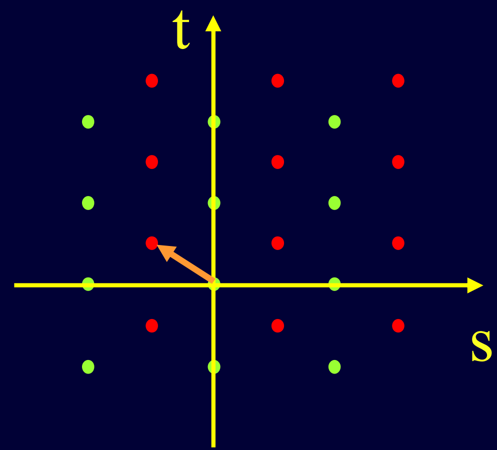


Angles $2k\phi$

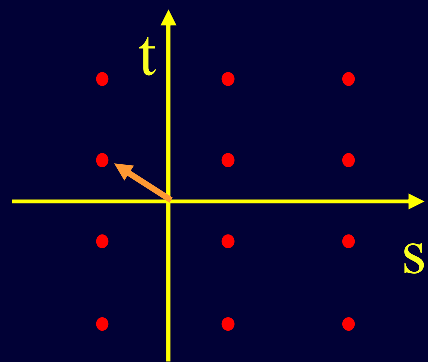



3D orthogonal grid

$$D = \frac{\pi}{b}\begin{pmatrix} 2\vartheta' & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{2}{\sqrt{3}} \end{pmatrix}$$

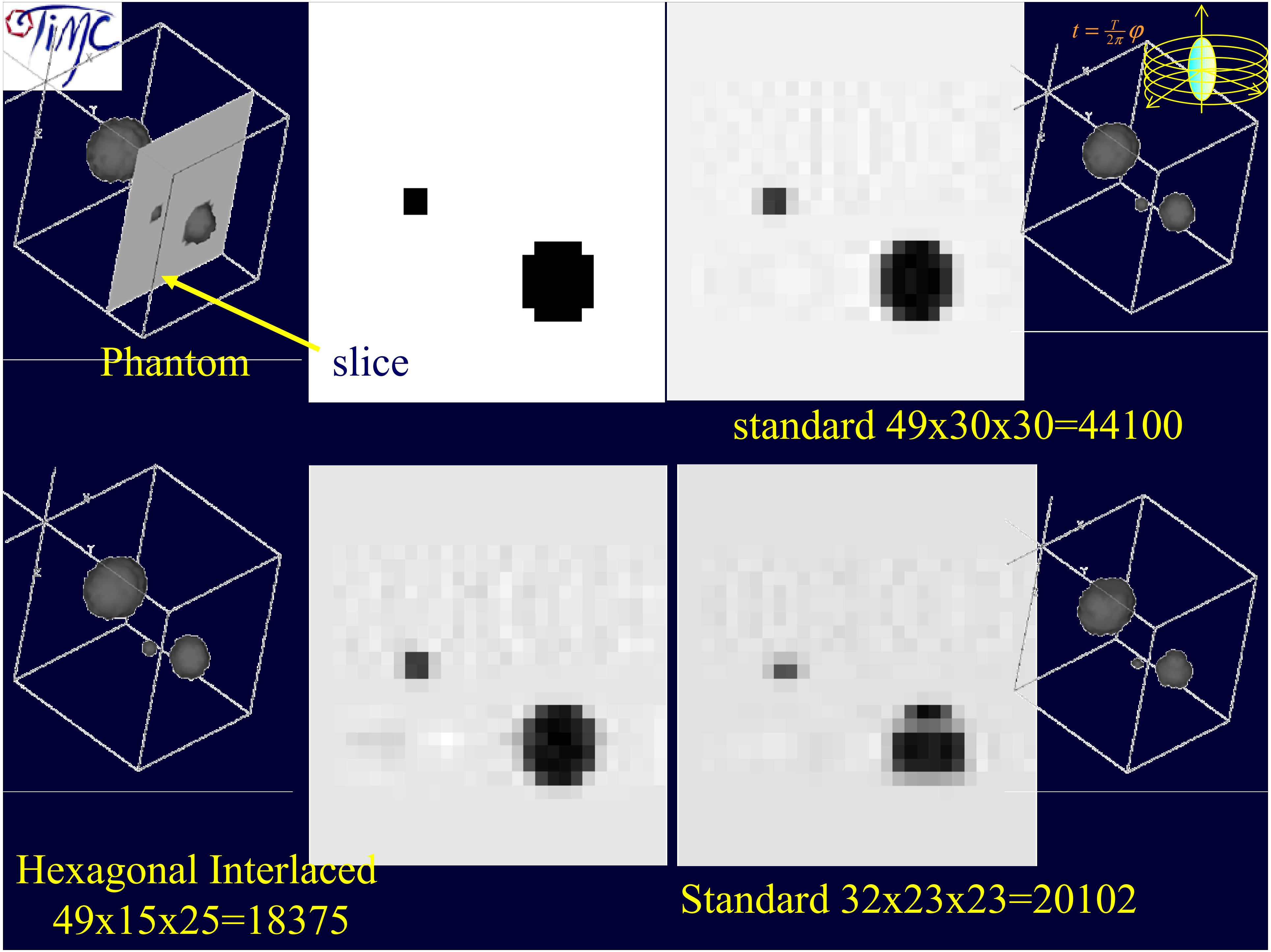



Angles $(2k+1)\phi$

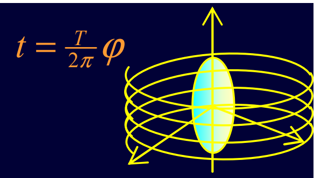



3D orthogonal grid

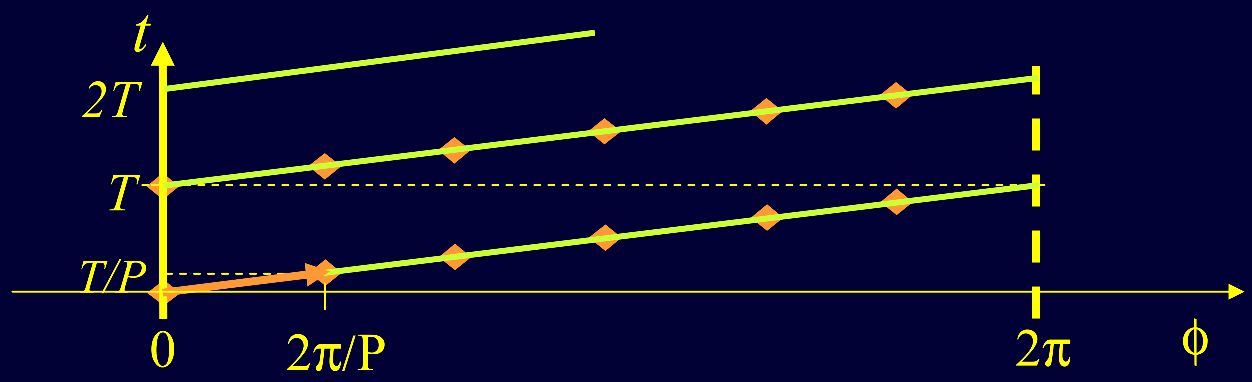
Apply Faridani 94 for new efficient schemes



$t = \frac{T}{2\pi}\varphi$
Phantom
slice
standard 49x30x30=44100
Hexagonal Interlaced
49x15x25=18375
Standard 32x23x23=20102




# Helical sampling



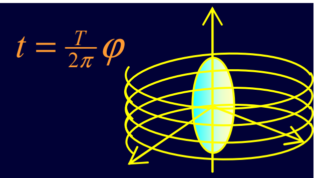

Representation of the lattice in the (ϕ,t) plane.

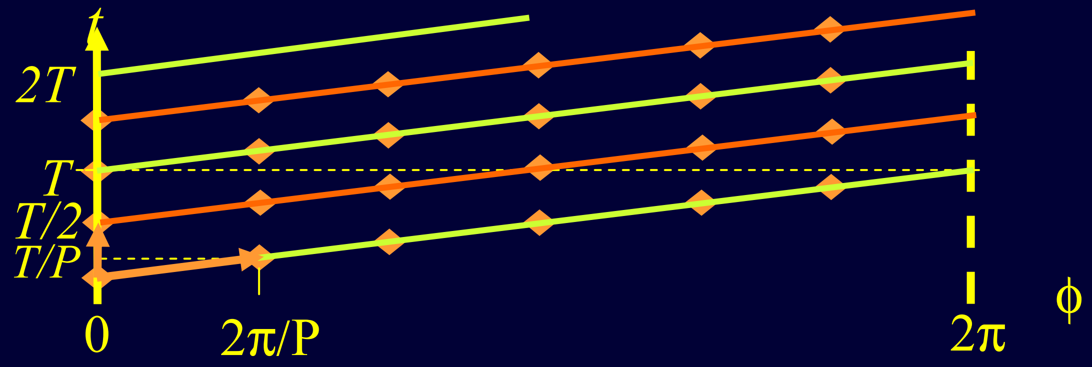
$$t(\varphi) = \frac{T}{2\pi}\varphi$$ Helical constraint

Periodicity

$$g(\varphi + 2\pi, s, t(\varphi + 2\pi)) = g(\varphi, s, t(\varphi + 2\pi))$$

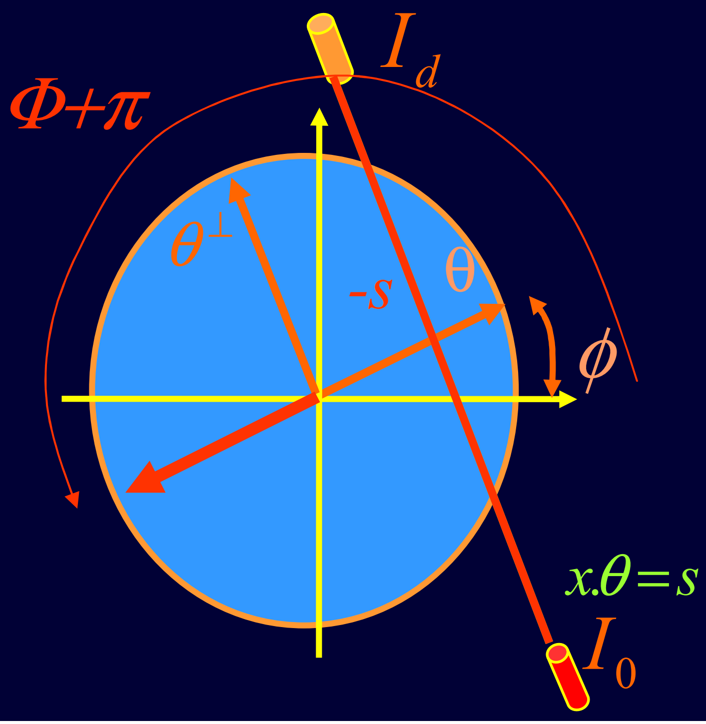
# Helical sampling



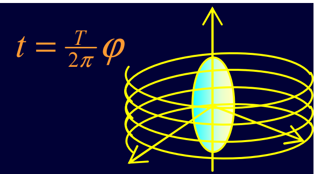




*P* even

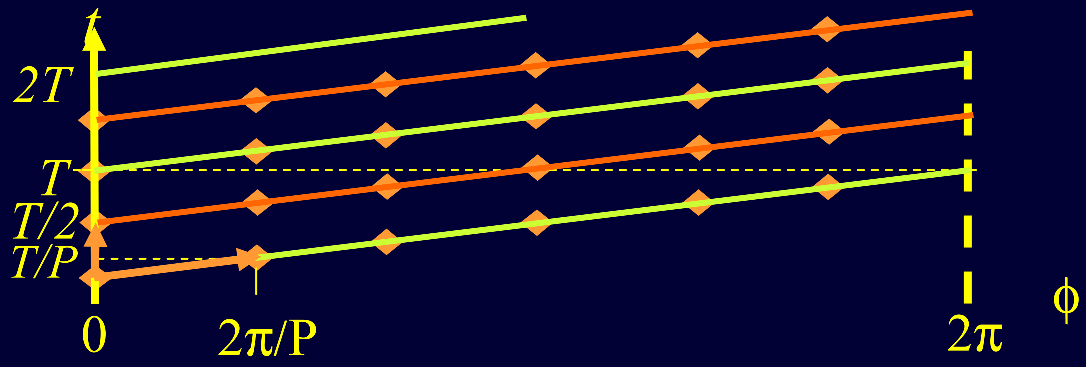
$$g(\varphi+\pi,-s,t(\varphi+\pi)) = g(\varphi,s,t(\varphi+\pi))$$




$$g(\phi,s) = g(\phi+\pi,-s)$$



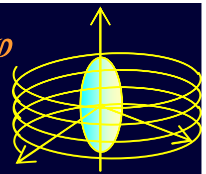

# Helical sampling



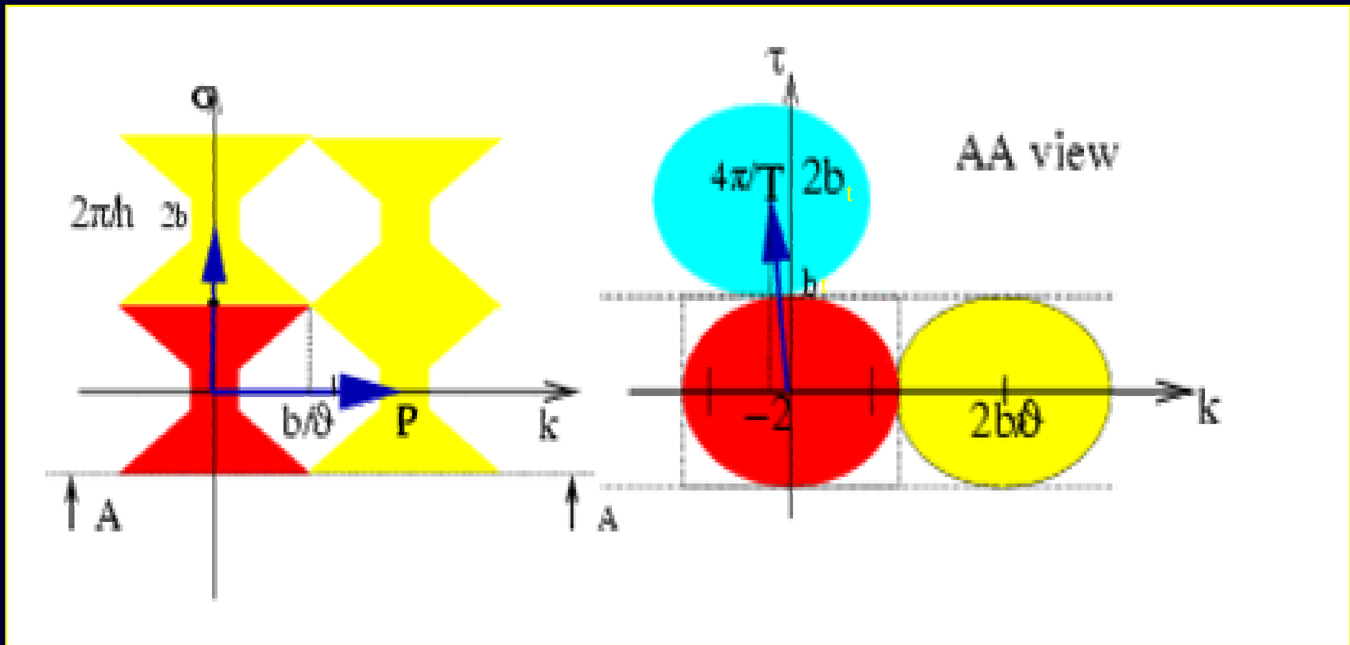

Representation of the lattice in the (ϕ,t) plane.
*P* has to be even.

$$W_{helS} = \begin{bmatrix} \frac{2\pi}{P} & 0 & 0 \\ 0 & h & 0 \\ \frac{T}{P} & 0 & \frac{T}{2} \end{bmatrix}$$

$t = \frac{T}{2\pi}\varphi$



# Helical sampling




$$2\pi W_{helS}^{-t} = \begin{bmatrix} P & 0 & -2 \\ 0 & 2\frac{\pi}{h} & 0 \\ 0 & 0 & 4\frac{\pi}{T} \end{bmatrix}$$

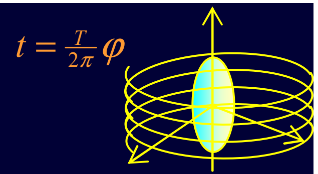
Shannon

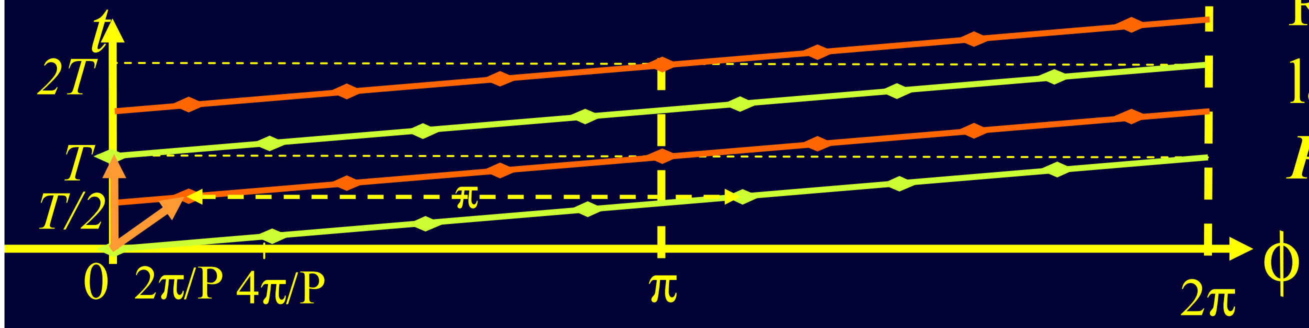
$$P \geq \frac{2b}{\vartheta}$$

$$h \leq \pi / b$$

$$T \leq 2\pi / b_t$$

# Efficient helical sampling

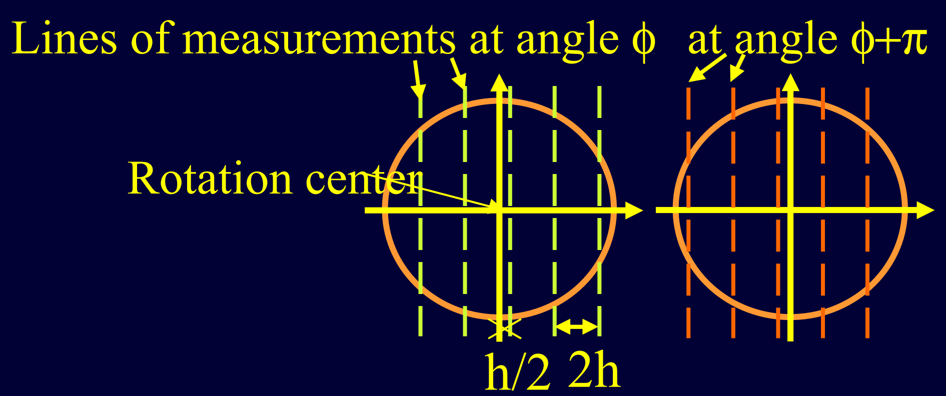



Representation of the lattice in the (ϕ,t) plane.
***P/2*** has to be odd.

- measured data
- data deduced from the symmetry property

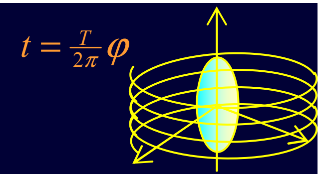
the green points and the red points are interlaced in the *s* direction



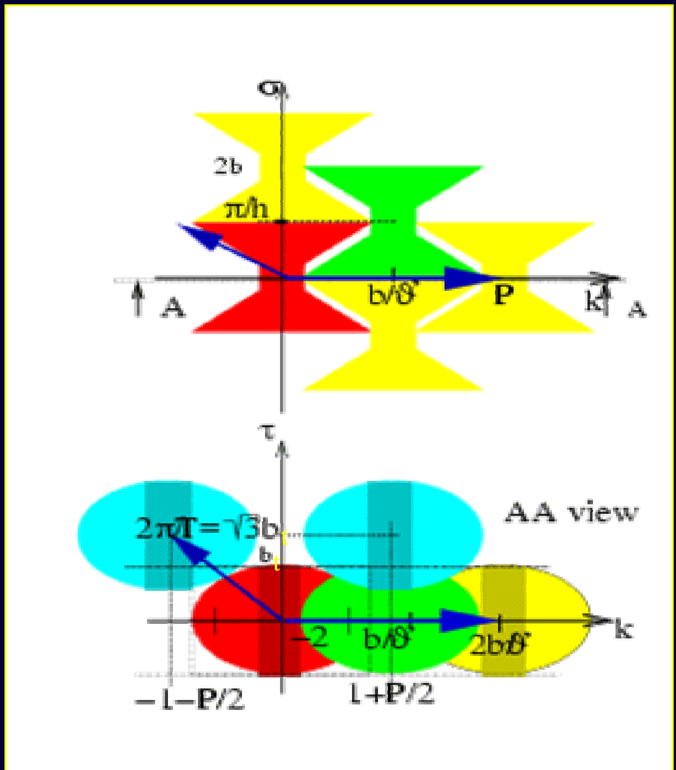

$$W_{helHI} = \begin{bmatrix} \frac{2\pi}{P} & 0 & 0 \\ h & 2h & 0 \\ \frac{T}{2} + \frac{T}{P} & 0 & T \end{bmatrix}$$

**Quater Detector Offset**

**La Rivière and Pan, 02**




# Helical Tomography Efficient // sampling



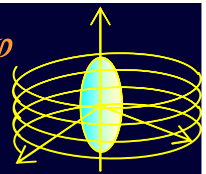

$$2\pi W_{helHI}^{-t} = \begin{bmatrix} P & -\frac{P}{2} & -\frac{P}{2}-1 \\ 0 & \frac{\pi}{h} & 0 \\ 0 & 0 & 2\frac{\pi}{T} \end{bmatrix}$$

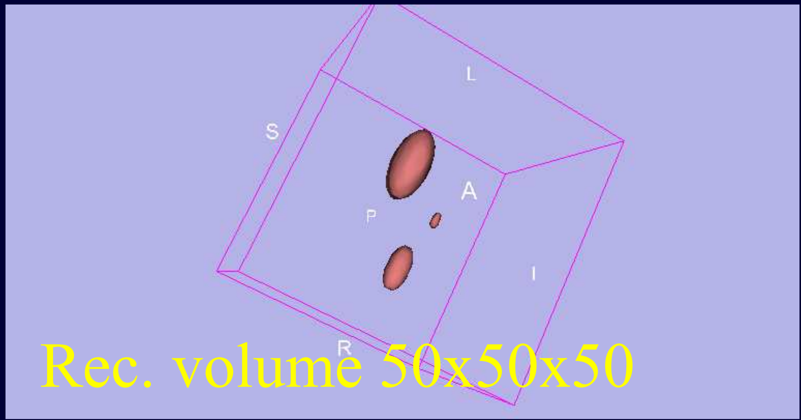
$$\text{Shannon:} \begin{cases} P \geq \frac{2b}{\vartheta''} \\ h = \frac{\pi}{b} \\ T = \frac{2\pi}{\sqrt{3}b_t} \end{cases}$$

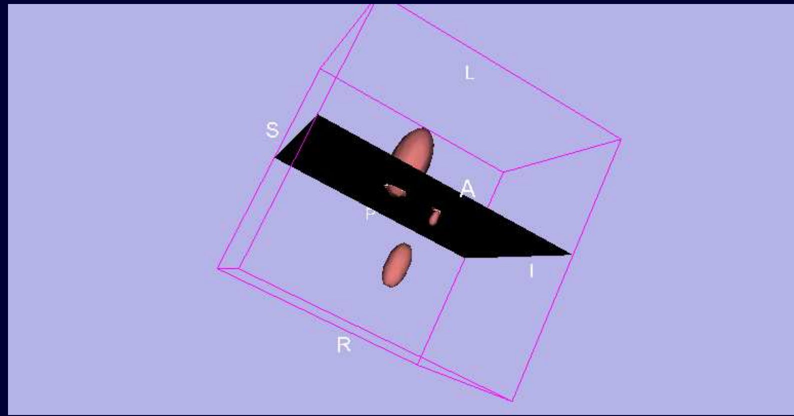
Sampling matrix and associated conditions

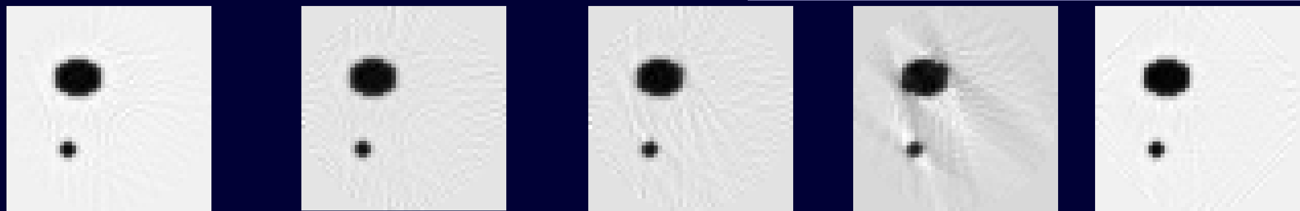
# Numerical results

$t = \frac{T}{2\pi}\varphi$



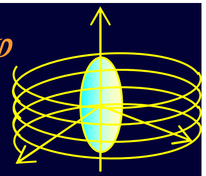


| 162S | 162I | 162HI | 126S | 214HI | |
|---|---|---|---|---|---|
| 25 | 25 | 44 | 20 | 57 | Turns |
| 162 | 162 | 81 | 126 | 107 | Projec. |
| 50 | 25 | 25 | 40 | 33 | Det.cells |
| 202500 | 101250 | 89100 | 100800 | 201267 | Data |
| 2,58 | 2,93 | 3,21 | 3,94 | 2,43 | $L^2$ error*100 |

$t = \frac{T}{2\pi}\varphi$



# Efficiency

$W_{helHI} / W_{helS}$

➔ $T_{rot}$ 1.73 times smaller

➔ **2 times less projections/turn**

➔ 2 times less detector cells

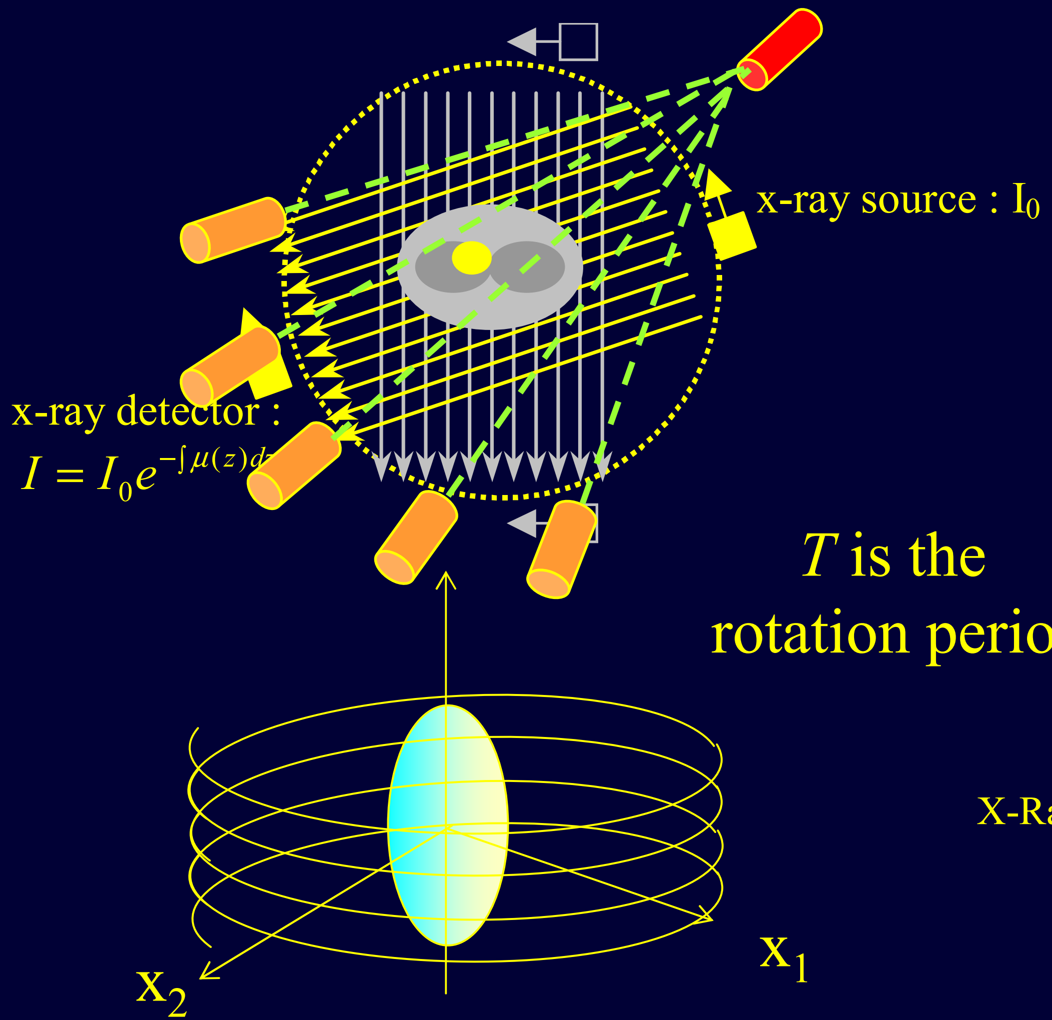
## => dose *1.73 / 2

## 15% de réduction

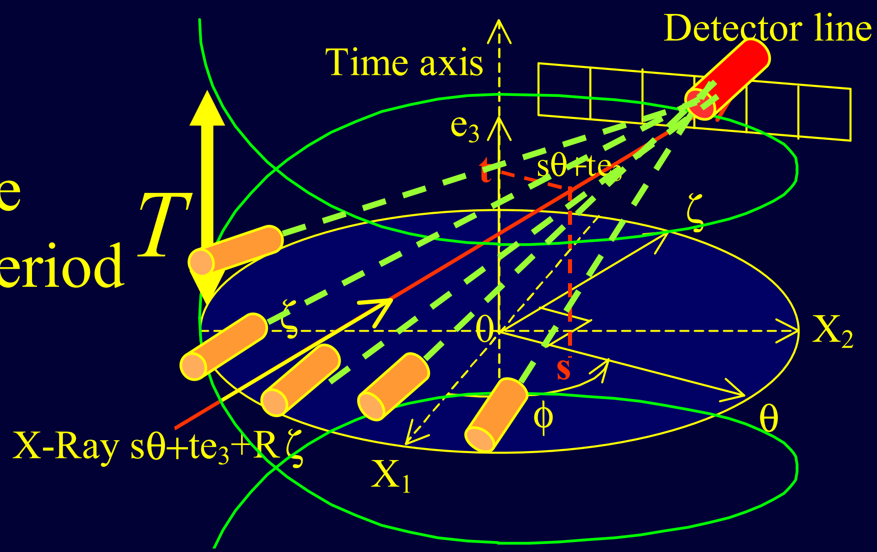
# Plan

- Tomography
- Sampling
- Sampling in tomography (Shannon approach)
- Sampling the Fan Beam X-ray Transform
- Efficient sampling in parallel helical tomography
- Discussion and perspectives

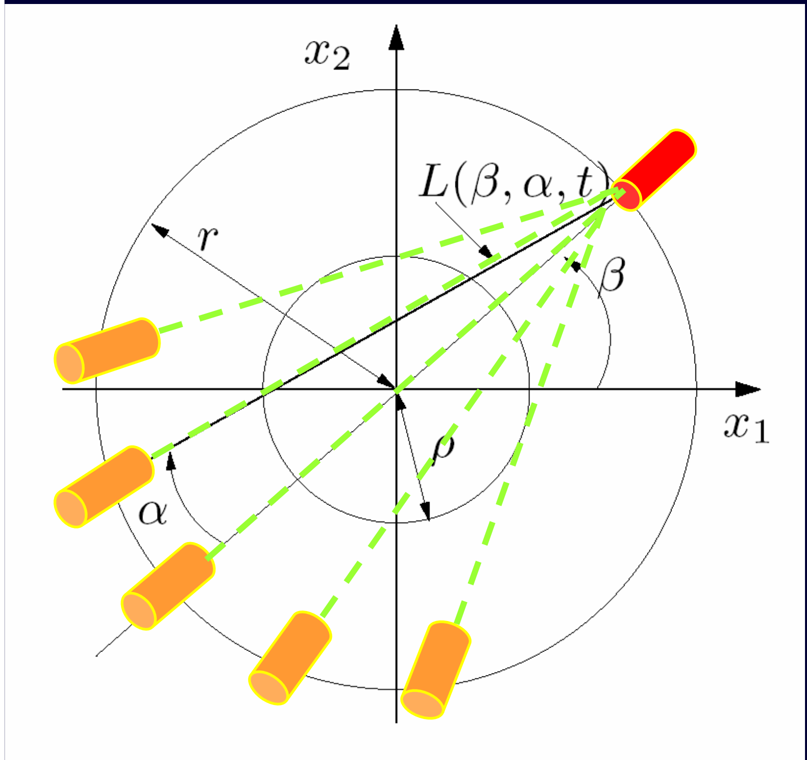
# Helical Fan Beam tomography




*T* is the rotation period



# Helical Fan Beam tomography

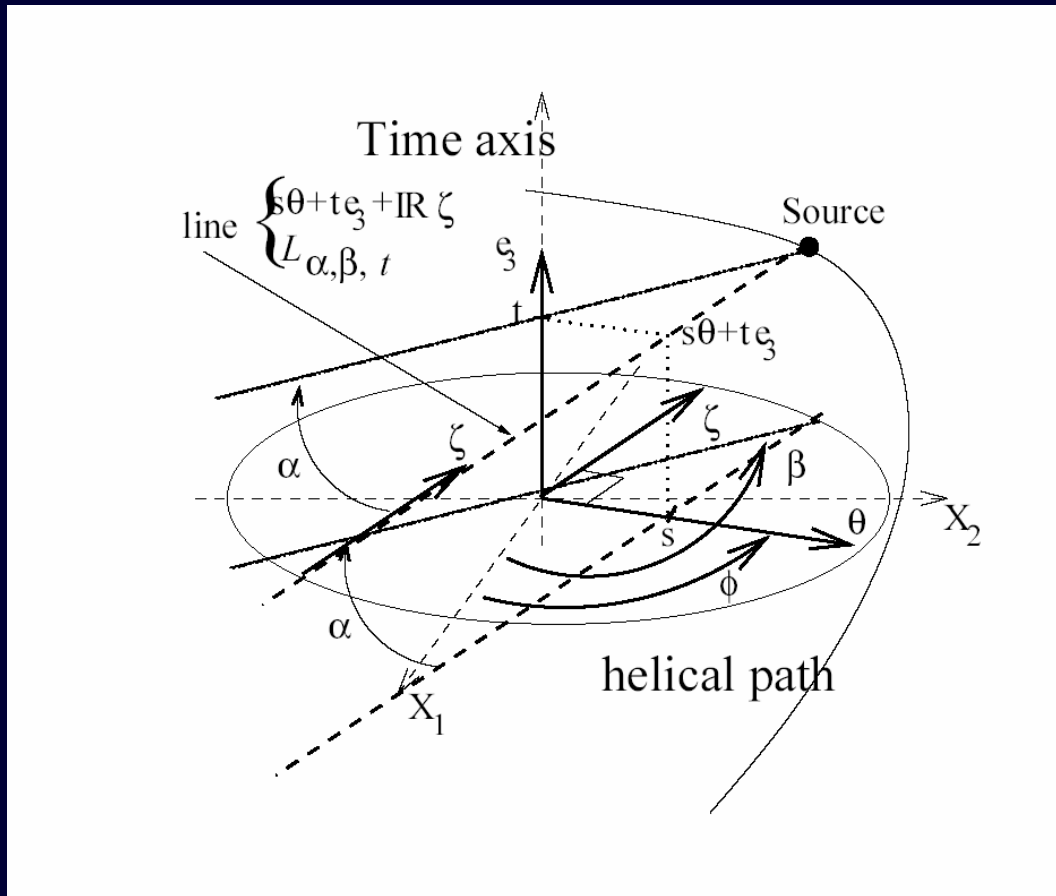



fan-beam geometry:

β : source angular position

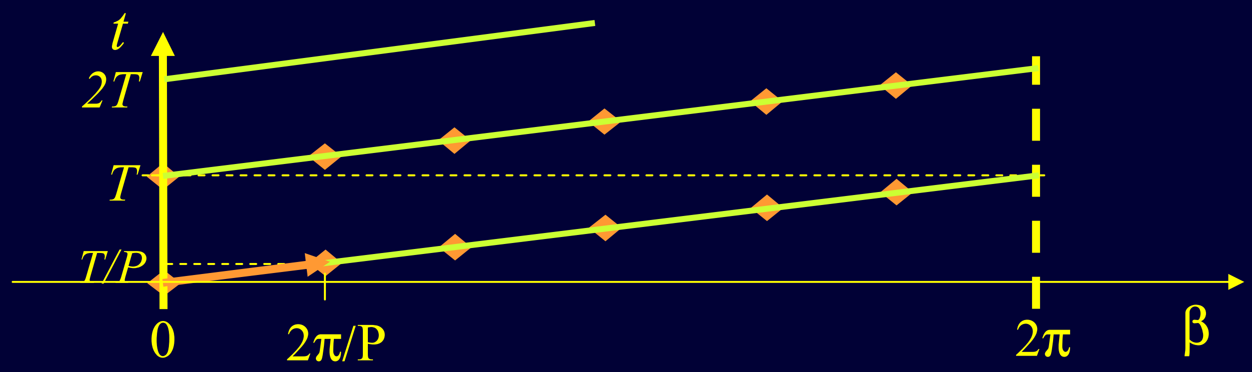
α : detector angular position

t : coordinate on the helix axis




Helical trajectory

# Fan beam helical tomography



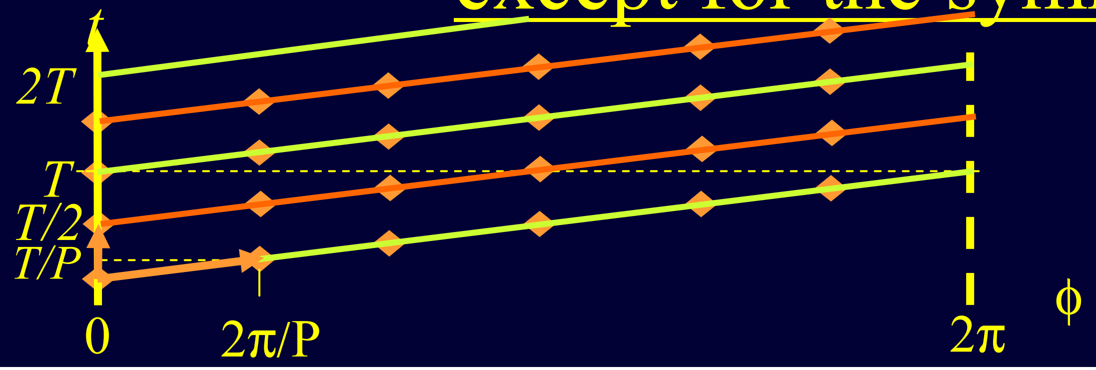

Lattice in the (β,t) plane.
*P* has to be even.

$$t(\beta) = \frac{T}{2\pi}\beta \quad \text{Helical constraint}$$

$$g(\beta + 2\pi, \alpha, t(\varphi + 2\pi)) = g(\beta, \alpha, t(\varphi + 2\pi))$$

# Apparently similar to the // case

# <u>except for the symmetry</u>




$$g(\beta,\alpha,t) = g(\beta+\pi+2\alpha,-\alpha,t) = g\left((A(\beta,\alpha,t)^t + (\pi,0,0)^t\right)$$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and } a = \begin{pmatrix} \pi \\ 0 \\ 0 \end{pmatrix}$$

If g is sampled on $L = \{Wk + \varepsilon, k \in \mathbf{Z}^3\}$, the symmetry yields

$$L_s = \{AWk + A\varepsilon + a, k \in \mathbf{Z}^3\}$$

Unfotunately $L \cup L_s$ , is generally not a lattice

**Natterer 99**

**Behmard, Faridani, 02**

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***END***