Large deviations for a zero mean asymmetric zero range process in random media.

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September 9, 2003

Abstract
We consider an asymmetric zero range process in infinite volume with zero mean and random jump rates starting from equilibrium. We investigate the large deviations from the hydrodynamical limit of the empirical distribution of particles and prove an upper and a lower bound for the large deviation principle. Our main argument is based on a super-exponential estimate in infinite volume. For this we extend to our case a method developed by Kipnis et al. (1989) and Benois et al. (1995).

Keywords: Asymmetric zero range, Hydrodynamical limit, Large deviations, Random environment.

AMS 2000 classification 60K35, 60K37, 82C22.

1 Introduction
Zero processes are a class of continuous time Markov processes with state space $X_d = (\mathbb{Z}_+)^d$. For $\xi \in X_d$ and $x \in \mathbb{Z}^d$, $\xi(x)$ represents the number
of particles at site $x$. Their dynamic can be informally described as follows: Each particle at site $x$ jumps, with a rate depending only on the total number of particles standing at this site, to a randomly chosen site $y$.

In this paper we consider a sequence of random variables $p = (p_x)_{x \in \mathbb{Z}^d}$, called an environment, in $[a_0, a_1]$ (where $0 < a_0 \leq a_1 < \infty$). According to $p$ the jump rates of the process will be accelerated or decelerated by the value $p_x$ at site $x$ (see (1) in Section 2).

Zero range processes in random or inhomogeneous media have been systematically and successfully investigated in the last few years: See for instance Benjamini et al. (1996), Evans (1996), Krug-Ferrari (1996), Landim (1996), Gielis et al. (1998), Bahadoran (1998), Seppäläinen-Krug (1999), Koukkous (1999), Andjel et al. (2000). We briefly review some of these results in the following lines.

Benjamini et al. (1996) have studied the asymmetric version of a zero-range process in infinite volume when the environment is an i.i.d. sequence of random variables (with $a_1 = 1$) and have proved the asymptotic hydrodynamical behavior of the system. Koukkous (1999) has investigated the hydrodynamical limit in the the symmetric case for a stationary and ergodic environment whose marginal law is absolutely continuous with respect to the Lebesgue measure. Indeed, thanks to Guo-Papanicolaou-Varadhan’s method (Guo et al. (1988)), he has shown that the empirical measure of particles converges in probability to the weak solution of a non-linear diffusion equation which does not depend on the environment $p$. The two main steps of his proof were to establish the so called one-block estimate and two-blocks estimate. Roughly speaking, the one-block estimate consists to show that the arithmetic mean of a local function in a large microscopic box can be replaced by its expectation with respect to the grand canonical measure with density equals to the average number of particles in the box. The two blocks estimate states
that the average number of particles in a large microscopic box is “close” to the average number in a small macroscopic box. The main difficulty was to overcome the non-translation invariance property of the invariant measures for the zero range process in random media (see (2)). Generalizing some ideas of Benjamini et al. (1996), Koukkous proved that one can specify some sites where the environment behaves badly in some ergodic sense allowing to conclude.

Following this investigation, the equilibrium fluctuations (Central limit results for the density field) were studied in Gielis et al. (1998). They proved that the density field converges weakly to a generalized Ornstein-Uhlenbeck process.

To continue the investigation around the hydrodynamical behavior a natural open question can be formulated as follows: Consider the hydrodynamical limit of the empirical measure with some continuous density \( \mu(\cdot) \) (with respect to Lebesgue measure) and let \( \Gamma \) be an event such that \( \mu \notin \bar{\Gamma} \). How to control the “deviant” behavior of the system inside \( \Gamma \)? This is the aim of a large deviation principle (LDP) for the hydrodynamical limit of the empirical measure.

This paper investigates this question for a \( d \)-dimensional zero mean asymmetric zero-range process in random media. In the deterministic case LDP have been obtained by many authors among which Landim (1992), Benois (1996) and Benois et al. (1995). In the last one Benois et al. showed an upper and a lower bound of the LDP in infinite volume for the empirical density when the process starts from equilibrium. The crucial ingredient of their arguments relies on the so-called super-exponential estimate: It consists to approximate, by some functions of the density field, the correlation field obtained by computing some exponential martingales related to the jumps of particles (as initiated in Kipnis et al. (1989) and Donsker-Varadhan (1989)). Once proved, this result implies, by standard arguments, the LDP (and also
the hydrodynamical limit) for the empirical measure.
In random environment the difficulty relies, once again, on the non-translation invariance property of the invariant measures of the process. For this reason we will also make use of the approach of Koukkous (1999).

The paper is organized as follows: In Section 2 notation and assumptions are stated as the main results. Section 3 is devoted to the proof of the super-exponential estimate. In the last section we prove the upper bound of the LDP. The proof of lower bound is omitted since, once the upper bound is proved, it relies on the arguments of Benois & al. (1995) without major modifications.

2 Notation and results

Let \(0 < a_0 \leq a_1 < \infty\) and consider a sequence of random variables \(\{p_x, \ x \in \mathbb{Z}^d\}\) on \([a_0, a_1]\) distributed according to an ergodic stationary measure \(m\), such that its one-dimensional marginal law is absolutely continuous with respect to the Lebesgue measure. We assume that \(m\{p : a_0 \leq p_0 \leq a_1\} = 1\) and for every \(\varepsilon > 0\), \(m\{p : p_0 \in [a_0, a_0 + \varepsilon]\}\) \(m\{p : p_0 \in (a_1 - \varepsilon, a_1]\} > 0\).

Denote by \(\mathbb{X}_d := (\mathbb{Z}^+)\mathbb{Z}^d\) the configuration space and by Greek letters \(\eta\) and \(\xi\) some of its typical elements. As usual \(\eta(x)\) stands for the total number of particles at site \(x \in \mathbb{Z}^d\) for the configuration \(\eta\).

For each given environment \(p\) we consider the Markov process \((\eta_t)_{t \geq 0}\) on \(\mathbb{X}_d\) whose generator is given by

\[
(\mathcal{L}_p f)(\eta) = \sum_{x, y \in \mathbb{Z}^d} p_x g(\eta(x)) T(x, y)[f(\eta^{x,y}) - f(\eta)],
\]

where \(f : \mathbb{X}_d \to \mathbb{R}\) is a bounded cylinder function, i.e. \(f\) only depends on \(\eta\) through a finite number of sites. \(T(\cdot, \cdot)\) is a transition probability on \(\mathbb{Z}^d\). The function \(g\) is positive and vanishes at 0: \(g(0) = 0 < g(k)\) for all \(k \geq 1\).
In the previous formula, \( \eta^{x,y}(z) \) is the configuration obtained from \( \eta \) when a particle jumps from \( x \) to \( y \):

\[
\eta^{x,y}(z) = \begin{cases} 
\eta(z) & \text{if } z \neq x, y \\
\eta(x) - 1 & \text{if } z = x \\
\eta(y) + 1 & \text{if } z = y 
\end{cases}
\]

For every non-negative real number \( \varphi \) we denote by \( \nu^\varphi \) the product measure on \( \mathbb{R}^d \) whose marginals are defined by

\[
\nu^\varphi \{ \eta : \eta(x) = k \} = \frac{1}{Z(\varphi p_x^{-1})} \frac{\varphi^k}{g(k)!}, \quad \text{for all } k \geq 0,
\]

where \( g(k)! = g(1)g(2)\ldots g(k) \) if \( k > 0 \) and \( g(0)! = 1 \). Under some hypotheses (see for instance [H1] and [H2] in what follows), those measures are invariant for the process (see Benjamini et al. (1996)). In this formula, \( Z : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \)

is the partition function

\[
Z(\varphi) = \sum_{k \geq 0} \frac{\varphi^k}{g(k)!}.
\]

Let \( \varphi^* \) be the radius of convergence of \( Z(\cdot) \); we assume that

\[
\lim_{\varphi \uparrow \varphi^*} Z(\varphi) = +\infty.
\]

Denote by \( \nu_\varphi(\cdot) := \nu^1_\varphi(\cdot) \) the invariant measure of the process \( (\eta_t)_{t \geq 0} \) when \( m \) is the Dirac measure concentrated on the set \( \{ p : p_x = 1, x \in \mathbb{Z}^d \} \) (see Andjel (1982)).

We define \( M : [0, \varphi^*) \rightarrow \mathbb{R}_+ \) by \( M(\varphi) = \nu_\varphi[\eta(0)] \), the expected number of particles at 0 with respect to \( \nu_\varphi \).

A simple computation shows that \( M(\varphi) = \varphi \partial_\varphi \log Z(\varphi) \) and from assumption (3) we check that \( M \) is an increasing, continuous, one-to-one function from \([0, \varphi^*) \) to \( \mathbb{R}_+ \).

We then define the “density” of particles (i.e. the expected number of particles at 0) with respect to the random media by the continuous and increasing
function \( R : [0, a_0 \varphi^*] \to \mathbb{R}^+ \) such that
\[
R(\varphi) = m[M(\varphi p_0^{-1})]
\]
and in order to ensure the existence of an invariant measure for any given value of the density, we assume that
\[
\lim_{\varphi \to a_0 \varphi^*} R(\varphi) = \infty.
\] (4)

Under this assumption the function \( R \) is one-to-one from \([0, a_0 \varphi^*]\) to \(\mathbb{R}_+\). We denote by \( \Phi \) its inverse (which is also continuous increasing bijection).

For a given density \( \rho > 0 \) we write
\[
\bar{\nu}_p^\rho = \nu_{p, \Phi(\rho)}.
\]

In the following we state all the hypotheses assumed throughout this paper.

[H1] The transition probability \( T(\cdot, \cdot) \) on \( \mathbb{Z}^d \) is a zero-mean irreducible translation invariant probability with finite range. That is
\[
T(x, y) = T(0, y - x) =: T(y - x),
\]
there exists a constant \( A > 0 \) such that \( T(x) = 0 \) if \( |x| \geq A \) and
\[
\sum_{x \in \mathbb{Z}^d} x T(x) = 0.
\]

[H2] The rate function \( g \) has bounded variation:
\[
g^* = \sup_k |g(k + 1) - g(k)| < \infty.
\]

Under the hypotheses [H1] and [H2] there exists a unique Markov process with corresponding generator defined by (1) for the deterministic case i.e. \( p \equiv 1 \) (see Andjel (1982)). Andjel’s proof applies also in the case we consider.
Let $(\sigma_{ij})_{\{1 \leq i,j \leq d\}}$ be a symmetric nonnegative definite matrix defined by the covariance matrix of the transition probability $T(\cdot)$:

$$
\sigma_{ij} = \sum_{y \in \mathbb{Z}^d} y_i y_j T(y) \quad \text{where} \quad y = (y_1, \cdots, y_d).
$$

[H3] In order to avoid some degenerate case of the hydrodynamic equation, we assume $(\sigma_{ij})_{\{1 \leq i,j \leq d\}}$ to be positive definite. That is there exists $\kappa > 0$ such that

$$
\sum_{i,j} \sigma_{ij} x_i x_j \geq \kappa \sum_i x_i^2, \quad \text{for all} \quad x = (x_1, \cdots, x_d) \in \mathbb{R}^d.
$$

[H4] To ensure some finite exponential moments of $\eta(x)$ under the measures $\nu_\varphi$, we shall assume that there exists a convex and increasing function $\omega : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

(i) $\omega(0) = 0$,

(ii) $\lim_{x \to \infty} (\omega(x)/x) = \infty$ and

(iii) for all density $\varphi$ there exists a positive constant $\theta := \theta(\varphi)$ such that

$$
\nu_\varphi \left[ \exp \{ \theta \omega(\eta(0)) \} \right] < \infty.
$$

This last assumption ensures also that $Z(\cdot)$ has infinite radius of convergence. It holds for example if $g(k+1) - g(k) \geq g_0^*$ for some constant $g_0^*$ and $k$ sufficiently large.

We will denote by $\omega^*$ the Legendre transform of $\omega$ given by:

$$
\omega^*(x) = \sup_{\alpha > 0} \{ \alpha x - \omega(\alpha) \}.
$$

In the next paragraphs, we define the state space of the process and its topology. Denote by $\mathcal{C}(\mathbb{R}^d)$ (resp. $\mathcal{C}_K(\mathbb{R}^d)$) the space of continuous (resp. with compact support) functions on $\mathbb{R}^d$ with classic uniform norm. Let $\mathcal{M}_+$ denote the space of positive Radon measures on $\mathbb{R}^d$ with the weak topology induced by $\mathcal{C}_K(\mathbb{R}^d)$ via $\langle \pi, H \rangle := \int H \, d\pi$ for $H \in \mathcal{C}_K(\mathbb{R}^d)$ and $\pi \in \mathcal{M}_+$. 


Fix a positive time parameter $T > 0$. For each realization of the environment $p$ and all fixed positive density $\rho$, $\mathbb{P}_\rho$ will denote the (quenched) probability measure on the path space $D([0, T], \mathbb{X}_d)$ corresponding to the Markov process $(\eta_t)_{t \in [0, T]}$ with accelerated generator $N^2 \mathcal{L}_p$ starting from the initial measure $\nu^\rho_{\rho}$. Denote by $\mathbb{E}^N_{\rho,\rho}$ the expectation under $\mathbb{P}^N_{\rho,\rho}$.

Let $\pi^N$ be the empirical measure associated to the particle system defined as

$$
\pi^N_t(du) = \pi^N(\eta_t, du) := \frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} \eta_t(x) \delta_{x/N}(du),
$$

for $0 \leq t \leq T$, where $\delta_r$ is the Dirac measure on $r \in \mathbb{R}$. Recall (see Kipnis-Landim (1999)) that there is a one to one correspondence between configurations $\eta$ and the empirical measure $\pi^N(\eta, du)$. Let $Q^N_{\rho,\rho}$ be the measure on the path space $D([0, T], \mathcal{M}_+)$ associated to the process $\pi^N$ with generator $N^2 \mathcal{L}_p$ starting from $\nu^\rho_{\rho}$.

To investigate the large deviations of the empirical measure we shall consider some small perturbations of the zero range process. To this purpose we introduce the following notation.

Let $\mathcal{C}^\ell_k([0, T] \times \mathbb{R}^d)$ denote the space of compact support functions with $\ell \in \mathbb{N}$ continuous derivatives in time and $k \in \mathbb{N}$ continuous derivatives in space.

For some smooth function $H$ in $\mathcal{C}^{1,2}_K([0, T] \times \mathbb{R}^d)$ we consider the perturbed Markov process generated by

$$
N^2(\mathcal{L}^{p,H}_{N,t} f)(\eta) = N^2 \sum_{x,y \in \mathbb{Z}^d} p_x g(\eta(x)) T(y) e^{\{H(t, \frac{x+y}{N}) - H(t, \frac{x}{N})\}} [f(\eta^{x,x+y}) - f(\eta)],
$$

where $f$ is a cylinder function. Denote $\mathcal{C}_\rho(\mathbb{R}^d)$ the set defined by

$$
\mathcal{C}_\rho(\mathbb{R}^d) = \mathcal{C}(\mathbb{R}^d) \cap \{ u : \mathbb{R}^d \to \mathbb{R}^+ ; \ u(x) = \rho \text{ for } |x| \text{ sufficiently large} \}.
$$
For a fixed $\gamma$ in $C_\rho(\mathbb{R}^d)$, let $\bar{\nu}^p_{\gamma,N}$ be the initial product measure of this process with marginals

$$\bar{\nu}^p_{\gamma,N}\{\eta, \eta(x) = k\} = \bar{\nu}^p_{\gamma(x/N)}\{\eta, \eta(x) = k\}$$

for all $x \in \mathbb{Z}^d$ and $k \in \mathbb{N}$. We therefore denote by $P^p_{\gamma,N}$ and $Q^p_{\gamma,N}$ the small perturbations of $P^N_{\rho,p}$ and $Q^N_{\rho,p}$ respectively.

For any path $\pi, D([0, T], \mathcal{M}_+)$ denote by $u_t$ its Radon-Nikodym derivative with respect to the Lebesgue measure $\lambda$: i.e. $u_t := \frac{d\pi_t}{d\lambda}$. Let $A = A(\rho)$ be the path (sub)space of $\pi, D([0, T], \mathcal{M}_+)$ such that $u_t$ is the solution of the PDE (Perturbed Heat Equation?) (which does not depends on the environment $p$)

\[
\begin{aligned}
\partial_t u &= \frac{\sigma}{2} \Delta (\Phi(u)) - \sum_{i=1}^{d} \partial_{x_i} (\Phi(u) \partial_{x_i} H) \\
\quad u(0, \cdot) &= \gamma(\cdot)
\end{aligned}
\]

for some initial profile $\gamma \in C_\rho(\mathbb{R}^d)$ and some $H \in C^{1,3}_K([0, T] \times \mathbb{R}^d)$. As usual $\Delta$ stands for the Laplacian operator.

The following notation is devoted to the definition of the rate functional of the LDP for $(\pi^N_t)_{0 \leq t \leq T}$.

For $H \in C^{1,2}_K([0, T] \times \mathbb{R}^d)$, we define $J_H : D([0, T], \mathcal{M}_+) \to \mathbb{R} \cup \{\infty\}$ by

$$J_H(\pi) = J^1_H(\pi) - J^2_H(\pi)$$

where

$$J^1_H(\pi) = \left\langle u_T, H_T \right\rangle - \left\langle u_0, H_0 \right\rangle - \int_0^T \left\langle u_t, \partial_t H_t \right\rangle \, dt,$$

$$J^2_H(\pi) = \frac{\sigma}{2} \int_0^T \left\langle \Phi(u_t), \sum_{i=1}^{d} \left( \partial_{x_i}^2 H_t + (\partial_{x_i} H_t)^2 \right) \right\rangle \, dt,$$

such that $J_H(\cdot) = +\infty$ outside $D([0, T], \mathcal{M}_+)$ or if $\pi_t$ is not absolutely continuous with respect to the Lebesgue measure $\lambda$ for some $0 \leq t \leq T$.  


We are now ready to define $I_0 : \mathcal{D}([0, T], \mathcal{M}_+) \to [0, \infty]$ the part of the large deviations rate function coming from the stochastic evolution:

$$I_0(\pi) = \sup_{H \in \mathcal{C}_k^1([0, T] \times \mathbb{R}^d)} J_H(\pi).$$

The other part of the large deviations rate function coincides with the behavior of deviations coming from the initial state: Let $h(\cdot|\rho)$ be the entropy defined for a positive function $\gamma : \mathbb{R}^d \to \mathbb{R}^+$ by

$$h(\gamma|\rho) = \int_{\mathbb{R}^d} \left\{ \gamma(x) \log \left( \frac{\Phi(\gamma(x))}{\Phi(\rho)} \right) - \mathbb{E}_m \left[ \log \left( \frac{Z(\Phi(\gamma(x))p_0^{-1})}{Z(\Phi(\rho)p_0^{-1})} \right) \right]\right\} \, dx.$$

JE NE COMPRENDS PAS. POURQUOI FAUT-IL PRENDRE L’ESPÉRANCE $\mathbb{E}_m$? N’EST-ON PAS EN MILIEU $p$ FIXE? (QUENCHED). CELA RESEMBLE PLUTÔT À UN RÉSULTAT ANNEALED...

D’AUTRE PART IL SERAÎT JUDICIEUX D’ÉCRIRE ICI UNE PETITE PHRASE DE COMMENTAIRE QUI COMPARE AVEC LE CAS DETERMINISTE DE Benois et al. COMME LE SUGGÈRE L’UN DES REFEREES.

Thus, the rate function of the large deviation principle is defined for a density $\rho > 0$ by

$$I_\rho(\pi) = I_0(\pi) + h(u_0|\rho).$$

From now on, for each $x \in \mathbb{Z}^d$, we denote by $\eta^l(x)$ the mean density of particles in a box of length $(2l + 1)$ centered at $x$:

$$\eta^l(x) = \frac{1}{(2l + 1)^d} \sum_{|y-x| \leq l} \eta(y).$$

For each cylinder function $\Psi : \mathbb{X}_d \to \mathbb{R}$, we define

$$\tilde{\Psi}(\rho) := m\left[ \nu_{\Psi(\rho)}^{\rho}(\Psi) \right].$$

We say that $\Psi$ is a Lipschitz function if

$$\exists k_0 \in \mathbb{N} \text{ and } c_0 > 0 \text{ such that } \left| \Psi(\eta) - \Psi(\xi) \right| \leq c_0 \sum_{|x| \leq k_0} \left| \eta(x) - \xi(x) \right|,$$
for all $\eta$ and $\xi$ in $\mathbb{X}_d$.

Finally denote by $\tau_x$ the shift operator defined by $\tau_x\Psi(\eta(\cdot)) := \Psi(\tau_x\eta(\cdot))$ where $\tau_x\eta(y) = \eta(x + y)$.

We can now state the super-exponential estimate:

**Theorem 2.1** Let $\Psi$ be a cylinder Lipschitz function and $H \in C_{K^*}^0([0, T] \times \mathbb{R}^d)$. Under hypotheses [H1] to [H4], for all $\delta > 0$ we have

$$\lim_{\varepsilon \to 0} \lim_{N \to \infty} \frac{1}{N^d} \log \mathbb{P}_{\rho, p}^N \left[ \left| \int_0^T W_{N, \varepsilon}^{H, \Psi}(t, \eta_t) \, dt \right| > \delta \right] = -\infty$$

almost surely, where

$$W_{N, \varepsilon}^{H, \Psi}(t, \eta) = \frac{1}{N^d} \sum_x H(t, x/N) \left[ \tau_x \Psi(\eta) - \tilde{\Psi}(\eta^{\varepsilon N}(x)) \right].$$

This theorem is a crucial argument in the proof of the LDP:

**Theorem 2.2** Under hypotheses [H1] to [H4], for every closed subset $C$ and every open subset $O$ of $D([0, T], \mathcal{M}_\pm)$, we have

$$\limsup_{N \to \infty} \frac{1}{N^d} \log Q_{\rho, p}^N(C) \leq - \inf_{\pi \in C} \mathcal{I}_\rho(\pi)$$

and

$$\liminf_{N \to \infty} \frac{1}{N^d} \log Q_{\rho, p}^N(O) \geq - \inf_{\pi \in O \cap A} \mathcal{I}_\rho(\pi)$$

almost surely.

**Remarks**

Before starting to prove our results, we would like to mention some facts and claims that will be used and whose proofs will be omitted. For more details the reader is referred to Kipnis-Landim’s book (1999) and Benois et al. (1995).
From Lemma I.3.5 in Kipnis-Landim (1999), the function defined by \( \varphi \mapsto \nu_\varphi \) for \( \varphi > 0 \), is an increasing function (see also the proof of Lemma 4.3 in Benois et al.). Therefore, assumption [H4] implies that for a fixed environment \( p \) as defined in the beginning of the previous section, for all \( x \in \mathbb{Z}^d \) and \( \varphi > 0 \), there exists \( \theta := \theta(x, \varphi) > 0 \) such that
\[
\nu_\varphi^p \left[ \exp \{ \theta \omega(\eta(x)) \} \right] < \infty \quad m\text{-almost surely.}
\]

Assumption [H4] ensures that the function \( \omega^* \) defined by (5) is also a continuous convex function such that \( \omega^*(0) = 0 \).

A simple computation shows that from the second condition in [H4], for every \( \varepsilon > 0 \) the function \( \omega^{-1}(r) - \varepsilon r \) is negative for each \( r \geq C_2(\varepsilon) \), for some constant \( C_2(\varepsilon) \) that depends only on \( \varepsilon \).

From the definition of \( \omega \) in [H4], the function defined on \( \mathbb{R}^*_+ \) by \( \Omega(r) = \frac{\omega(r)}{r} \) is an increasing function.

For each cylinder Lipschitz function \( \Psi(\cdot) \), the function \( \tilde{\Psi}(\cdot) \) given by (6) is also a Lipschitz function (see Lemma I.3.6 of Kipnis-Landim (1999)). Moreover one can check that \( \tilde{\Psi}(k) \leq Ck \) for all \( k \in \mathbb{Z} \) for some constant \( C \).

The strategy we adopt to prove the results is similar to the one presented in Benois & al. (1995). However, we need some arguments developed in Koukkous (1999) in order to overcome the lack of translation invariance of the invariant measures for the zero range process in random media. We will thus focus only on the main differences.

From now on, to keep the notation simple, we will restrict our study to the one-dimensional case. The extension to higher dimension is straightforward.
3 Proof of Theorem 2.1

Let $G$ be a positive continuous function on $\mathbb{R}$ defined by

$$G(x) = \sup_{y \in [x-1, x+1]} \max \left\{ |H(y)|, |\partial_y H(y)|, |\partial_y^2 H(y)| \right\}. \quad (8)$$

We have

$$\mathbb{P}_N^\rho,p \left[ \int_0^T W^{H,\Psi}_{N,\varepsilon}(t, \eta_t) \, dt > \delta \right] \leq \mathbb{P}_N^\rho,p \left[ \int_0^T \left\{ W^{H,\Psi}_{N,\varepsilon}(t, \eta_t) \, dt - \frac{\beta}{N} \sum_x G \left( \frac{x}{N} \right) \omega(\eta_t(x)) \right\} \, dt > \delta/2 \right] + \mathbb{P}_N^\rho,p \left[ \int_0^T \frac{\beta}{N} \sum_x G \left( \frac{x}{N} \right) \omega(\eta_t(x)) \, dt > \delta/2 \right] \quad (9)$$

for every $\beta > 0$.

By Tchebycheff exponential inequality the first term in the left hand side in (9) is bounded above by

$$\exp \left\{ -N \theta \delta / 2 \right\} \mathbb{E}_N^\rho,p \left[ \exp \left( \theta \int_0^T \left\{ NW^{H,\Psi}_{N,\varepsilon}(t, \eta_t) - \beta \sum_x G \left( \frac{x}{N} \right) \omega(\eta_t(x)) \right\} \, dt \right]$$

for every $\theta > 0$.

Therefore, we have to prove two Lemmas:

**Lemma 3.1** For every $G \in C_K(\mathbb{R})$,

$$\lim_{A \to \infty} \lim_{N \to \infty} \frac{1}{N} \log \mathbb{P}_N^\rho,p \left[ \int_0^T \frac{1}{N} \sum_x G(x/N) \omega(\eta_t(x)) \, dt > A \right] = -\infty \quad (10)$$

$m$-almost surely.

**Lemma 3.2** For any $\theta > 0$ and $\beta > 0$

$$\lim_{\varepsilon \to 0} \lim_{N \to \infty} \frac{1}{N} \log \mathbb{E}_N^\rho,p \left[ \exp \theta \int_0^T \left\{ W^{H,\Psi}_{N,\varepsilon}(t, \eta_t) - \beta \sum_x G \left( \frac{x}{N} \right) \omega(\eta_t(x)) \right\} \, dt \right] = 0. \quad (11)$$

$m$-almost surely.
Proof of Lemma 3.1.

Using respectively Tchebycheff exponential inequality and Jensen inequality, we show that for every positive constant \( \theta \), the logarithmic term in (10) is bounded above by

\[-\theta AN + \log E_{\rho,p}^{N} \left[ \frac{1}{T} \int_{0}^{T} \exp \left\{ \sum_{x} \theta TG(x/N)\omega(\eta(x)) \right\} \, dt \right].\]

From the begining of [R1] and since the product measure \( \tilde{\nu}_{\rho} \) is invariant for the process and \( p_{x} \in [a_{0}, a_{1}] \), a simple computation shows that the left hand side term in (10) is bounded above by

\[
\lim_{A \to \infty} \lim_{N \to \infty} \inf_{\theta > 0} \left\{ -\theta A + \frac{1}{N} \sum_{x} \log \nu_{\Phi (\rho) a_{0}^{-1}} \left[ \exp \left\{ \theta T |G(x/N)|\omega(\eta(0)) \right\} \right] \right\}.
\]

Let \( B > 0 \) be such that

\[\text{supp} G \subset [-B, B].\]

From [H4], there exists \( \theta_{0} > 0 \) such that

\[\nu_{\Phi (\rho) a_{0}^{-1}} \left[ \exp \left\{ \theta_{0} T \|G\|_{\infty} \omega(\eta(0)) \right\} \right] < \infty.\]

The lemma is proved since (12) is bounded above by

\[
\lim_{A \to \infty} \left\{ -\theta_{0} A + 2B \log \nu_{\Phi (\rho) a_{0}^{-1}} \left[ \exp \left\{ \theta_{0} T \|G\|_{\infty} \omega(\eta(0)) \right\} \right] \right\}.
\]

Proof of Lemma 3.2.

Let

\[V(\eta) = \theta \left\{ NW_{N,\varepsilon}^{H,\Phi}(0, \eta) - \beta \sum_{x} G \left( \frac{x}{N} \right) \omega(\eta(x)) \right\} \]

Let \( \mathcal{L}_{V}^{p} \) be the generator \( N^{2} \mathcal{L}_{p} + V \) and \( \mathcal{L}_{V}^{p*} \) its adjoint operator, which is equal to \( N^{2} \mathcal{L}_{p}^{*} + V \). If we denote by \( S_{V}^{p} \) the semigroup associated to the generator \( \mathcal{L}_{V}^{p} \), by the Feynman-Kac formula the expectation in the lemma is equal to

\[\langle S_{T}^{V,p} 1, 1 \rangle \leq \langle S_{T}^{V,p} 1, S_{T}^{V,p} 1 \rangle^{\frac{1}{2}}.\]
Now, if we denote by $\lambda_V$ the largest eigenvalue of the self-adjoint operator $\mathcal{L}_V^p + \mathcal{L}_V^{p,*}$,
\[
\partial_t \langle S_t^{V,p}, S_t^{V,p} \rangle = \langle (\mathcal{L}_V^p + \mathcal{L}_V^{p,*}) S_t^{V,p}, S_t^{V,p} \rangle \leq \lambda_V \langle S_t^{V,p}, S_t^{V,p} \rangle.
\]

By Gronwall’s lemma we show that
\[
\langle S_T^{V,p}, S_T^{V,p} \rangle \leq \exp \left\{ T \lambda_V \right\}. \quad (13)
\]

Recall that we did not assume $T(\cdot)$ to be symmetric and therefore $\nu^p_\Phi(\rho)$ can be non-reversible for the process. However, at this level, our study is dealing with the reversible generator $N^2(\mathcal{L}_p + \mathcal{L}_p^*)$. Thus we can assume the generator $\mathcal{L}_p$ to be reversible and $T(\cdot)$ given by $T(x) = (1/2) 1_{\{|x|=1\}}$.

Let
\[
I_{p,x,x+1}^p(f) = \frac{1}{2} \int p_x g(\eta(x)) \left[ \sqrt{f(\eta^{x,x+1})} - \sqrt{\tilde{f}(\eta)} \right]^2 \tilde{\nu}_p^p(\,d\eta),
\]
and $D_p(\cdot)$ the Dirichlet form given by
\[
D_p(f) = \sum_x I_{p,x,x+1}^p(f).
\]

Using the variational formula for the largest eigenvalue of a self-adjoint operator (see appendix A3.1 of Kipnis-Landim (1999)), from (13) we reduce the proof of the lemma to show that for every positive $\theta$
\[
\lim_{\varepsilon \to 0} \lim_{N \to \infty} \sup_f \left\{ \int \left[ \theta \left( W_{N,\varepsilon}^H(\eta) - \frac{\beta}{N} \sum_x \tau_x(\eta(x)) \omega(\eta(x)) \right) f(\eta) \tilde{\nu}_p(\,d\eta) - N D_p(f) \right] \right\} \leq 0.
\]

The supremum is taken over all positive densities functions with respect to $\tilde{\nu}_p^p$.

We use now some computations from Benois & al. (1995) and Kipnis & al. (1989). Let
\[
W_t^\Psi(\eta) = \frac{1}{2l + 1} \sum_{|y| \leq l} \tau_y \Psi(\eta) - \bar{\Psi}(\eta(0))
\]
In this way, we can rewrite the term

\[ W_{N,\epsilon}^{H,\Psi}(\eta) - \frac{\beta}{N} \sum_x G\left(\frac{x}{N}\right) \omega(\eta(x)) \]

as

\[
\frac{1}{N} \sum_x \left\{ H\left(\frac{x}{N}\right) \left[ \tau_x \Psi(\eta) - \frac{1}{2l+1} \sum_{|y-x| \leq l} \tau_y \Psi(\eta) \right] - \frac{\beta}{3} G\left(\frac{x}{N}\right) \omega(\eta(x)) \right\} \\
+ \frac{1}{N} \sum_x \left\{ H\left(\frac{x}{N}\right) \tau_x W^{\Psi}(\eta) - \frac{\beta}{3} G\left(\frac{x}{N}\right) \omega(\eta(x)) \right\} \\
+ \frac{1}{N} \sum_x \left\{ H\left(\frac{x}{N}\right) \left[ \Psi(\eta(x)) - \Psi(\eta^{\epsilon N}(x)) \right] - \frac{\beta}{3} G\left(\frac{x}{N}\right) \omega(\eta(x)) \right\}.
\]

From the assumption on \( \Psi \), we check easily that there exists \( C(\Psi, a_0, a_1) \) such that for all \( x \in \mathbb{Z} \), \( \Psi(\eta(x)) \leq C(\Psi, a_0, a_1) \eta(x) \). Then from the definitions of \( \omega^*(\cdot) \) and \( G(\cdot) \) (cf. (5) and (8)), the first term in the last expression is bounded above by

\[
\frac{1}{N} \sum_x \left\{ \frac{1}{2l+1} \sum_{|y-x| \leq l} H\left(\frac{y}{N}\right) - H\left(\frac{x}{N}\right) \right\} \left[ \Psi(\eta(x)) - \frac{\beta}{3} G\left(\frac{x}{N}\right) \omega(\eta(x)) \right] \leq \frac{\beta}{3N} \sum_x G\left(\frac{x}{N}\right) \frac{3C(\Psi, a_0, a_1)l}{\beta N} \eta(x) - \omega(\eta(x)) \right\} \\
\leq \omega^* \left\{ \frac{3C(\Psi, a_0, a_1)l}{\beta N} \right\} \frac{\beta \|G\|_{\infty}}{3}.
\]

This last term vanishes as \( N \uparrow \infty \) since \( \omega^*(\cdot) \) is continuous and \( \omega^*(0) = 0 \).

Now, to achieve the proof of Lemma 3.2, we shall prove:

**Lemma 3.3** For any \( b > 0 \) and \( \beta > 0 \)

\[
\lim_{l \to \infty} \lim_{N \to \infty} \sup_f \left\{ \frac{1}{N} \sum_x \int \left[ H\left(\frac{x}{N}\right) \tau_x W^{\Psi}(\eta) - \beta G\left(\frac{x}{N}\right) \omega(\eta(x)) \right] f(\eta) \ d\nu^p(\eta) - b N D_p(f) \right\} \leq 0
\]

\( m \)-almost surely. The supremum is taken over all positive densities functions with respect to \( \nu^p \).
And, thanks to remarks [R5], we have to prove that:

**Lemma 3.4** For any $b > 0$ and $\beta > 0$

\[
\lim_{l \to \infty} \lim_{\varepsilon \to 0} \lim_{N \to \infty} \sup_{f} \left\{ \frac{1}{N} \sum_{x} \int \left[ H\left( \frac{x}{N} \right) \left| \eta^{N}(x) - \eta(x) \right| - \beta G\left( \frac{x}{N} \right) \omega(\eta(x)) \right] f(\eta) \, d\nu_{\rho}^{p}(\,d\eta) - bN D_{p}(f) \right\} \leq 0
\]

(15)

$m$-almost surely. The supremum is taken over all positive densities functions with respect to $\nu_{\rho}^{p}$.

**Proof of Lemma 3.3.**

Using the convexity of $\omega$ and definition of $G$, we check that

\[
\frac{1}{N} \sum_{x} H\left( \frac{x}{N} \right) \omega(\eta(x)) \leq \frac{1}{N} \sum_{x} H\left( \frac{x}{N} \right) \left| \frac{1}{2l + 1} \sum_{|y-x| \leq l} \omega(\eta(y)) \right|
\]

\[
= \frac{1}{N} \sum_{x} \omega(\eta(x)) \left| \frac{1}{2l + 1} \sum_{|y-x| \leq l} H(y/N) \right|
\]

\[
\leq \frac{1}{N} \sum_{x} \omega(\eta(x)) G\left( \frac{x}{N} \right)
\]

(16)

We first introduce some notation in order to deal in our study of (14) with the boxes of length $(2l + 1)$. Indeed, the term

\[
H\left( \frac{x}{N} \right) \tau_{x} W_{t}^{p}(\eta) - \beta H\left( \frac{x}{N} \right) \omega(\eta(x))
\]

depends on $\eta$ only through $\eta(x - l) \cdots \eta(x + l)$. Thus we may restrict the integral to microscopic blocks. Denote by $\Lambda_{l} = \{-l \cdots l\}$ the box of length $(2l + 1)$ centered at the origin. For a fixed $z \in \mathbb{Z}$, we denote by $\Lambda_{z,l}$ the box $z + \Lambda_{l}$, by $\mathbb{X}^{l}$ the configuration space $\mathbb{N}^{\Lambda_{l}}$, by $\nu_{\rho,z,l}^{p}$ the product measure $\nu_{\rho}^{p}$ restricted to $\mathbb{X}^{l}$, by $f_{z,l}$ the density, with respect to $\nu_{\rho,z,l}^{p}$, of the marginal of the measure $f(\eta) \nu_{\rho}^{g,p}(d\eta)$ on $\mathbb{X}^{l}$ and by $D_{p,z,l}(h)$ the Dirichlet form on $\mathbb{X}^{l}$ given by

\[
D_{p,z,l}(h) = \sum_{|x-y|=1} \int_{x \in \Lambda_{z,l}} \int_{y \in \Lambda_{z,l}} p_{x,y}(\eta(x)) \left[ \sqrt{h(\eta^{y})} - \sqrt{h(\eta)} \right]^{2} \nu_{\rho,z,l}^{p}(\,d\eta).
\]
Thus, from (16) and since the Dirichlet form is convex (by Schwarz inequality), the supremum in the lemma is bounded above by the supremum over all positive densities \( f \) (with respect to \( \bar{\nu}^\rho \)) of the term

\[
\frac{1}{N} \sum_x \left\{ \int \left(\frac{x}{N}\right) W_l^q(\eta) 1_{\{\eta'(0) \leq A\}} f_{x,l} \bar{\nu}_\rho \, d\eta - \frac{b N^2}{C(l)} D^p_{\rho,x,l}(f_{x,l}) \right\}
\]

\[
+ \frac{1}{N} \sum_x \left| H\left(\frac{x}{N}\right) \right| \left\{ \int \left[ W_l^q(\eta) - \beta \omega(\eta'(0)) \right] 1_{\{\eta'(0) \geq A\}} f_{x,l} \bar{\nu}_\rho \, d\eta \right\}
\]

The first term can be treated in the same way as the one of formula (3) in Koukkous (1999). Let

\[
\mathbb{E}^P_f[h] = \int h(\eta) f(\eta) \, d\bar{\nu}^P_{\rho,0,l}(\eta),
\]

\[
\mathcal{B}^P_l(A) = \left\{ f \in \mathcal{B}^P_l : \mathbb{E}^P_f[\eta'(0)] \geq A \right\}.
\]

and \( B > 0 \) be such that \( supp H \subset [-B, B] \). Thus, the supremum over all positive densities \( f \) (with respect to \( \bar{\nu}^P_\rho \)) of the second term in the last expression is bounded above by

\[
2(2B+1)C(\Psi) \|H\|_\infty \sup_p \sup_{f \in \mathcal{B}^P_l(A)} \left\{ \mathbb{E}^P_f[\eta'(0)] - \frac{\beta}{2C(\Psi)} \mathbb{E}^P_f[\omega(\eta'(0))] \right\}
\]

Recall that \( \omega \) is a convex and increasing function. Thus, by Jensen’s inequality, the last expression is bounded above by

\[
2(2B+1)C(\Psi) \|H\|_\infty \sup_p \sup_{f \in \mathcal{B}^P_l(A)} \left\{ \omega^{-1}\left[ \mathbb{E}^P_f[\omega(\eta'(0))] \right] - \frac{\beta}{2C(\Psi)} \mathbb{E}^P_f[\omega(\eta'(0))] \right\}.
\]

From the Remark [R3], we claim that there exists a finite constant \( C_2 = C_2(\beta, C(\Psi)) \) such that this last expression is non positive term if \( \mathbb{E}^P_f[\eta'(0)] \geq C_2 \). To conclude the proof, we take \( A \) larger than \( C_2 \).
Proof of Lemma 3.4.
First of all, we approximate (replace) the average over a small macroscopic box by an average over large microscopic boxes. More precisely, for \( N \) sufficiently large we check that

\[
\frac{1}{N} \sum_x |H\left(\frac{x}{N}\right)| \left| \eta^{\varepsilon N}(x) - \eta^l(x) \right| \leq \frac{1}{N} \sum_x \left| H\left(\frac{x}{N}\right) \right| \left( \frac{1}{(2\varepsilon N + 1)} \sum_{2l+1 < |y| \leq \varepsilon N} |\eta^l(x) - \eta^l(x + y)| + \mathcal{O}\left( \frac{1}{\varepsilon N^2} \right) \sum_x G\left(\frac{x}{N}\right) \eta(x) \right)
\]

\[
\leq \frac{1}{N} \sum_x \left| H\left(\frac{x}{N}\right) \right| \left( \frac{1}{(2\varepsilon N + 1)} \sum_{2l+1 < |y| \leq \varepsilon N} |\eta^l(x) - \eta^l(x + y)| + \frac{\beta}{N} \sum_x G\left(\frac{x}{N}\right) \omega(\eta(x)) \right)
\]

For \( A > 0 \), define

\[
W_{\varepsilon,N}^{\varepsilon,N}(\eta, x) = \frac{1}{(2\varepsilon N + 1)} \sum_{2l+1 < |y| \leq \varepsilon N} |\eta^l(x) - \eta^l(x + y)| 1_{\{\eta^l(x) \lor \eta^l(x+y) \leq A\}}
\]

and

\[
\overline{W}_{\varepsilon,N}^{\omega}(\eta, x, y) = \left[ \left| \eta^l(x) - \eta^l(x+y) \right| - \left( \omega(\eta^l(x)) + \omega(\eta^l(x+y)) \right) \right] 1_{\{\eta^l(x) \lor \eta^l(x+y) \geq A\}}
\]

Thus, we show that, for \( N \) and \( A \) sufficiently large, the term between braces in (15) is bounded above by

\[
\left\{ \frac{1}{N} \sum_x \int \left| H\left(\frac{x}{N}\right) \right| W_{\varepsilon,N}^{\varepsilon,N}(\eta, x) f(\eta) \bar{\nu}_p^\rho(\, d\eta) \right\} - bND_p(f)
\]

\[
+ \frac{1}{N} \sum_x \int \left[ \left| H\left(\frac{x}{N}\right) \right| \left( \frac{1}{(2\varepsilon N + 1)} \sum_{2l+1 < |y| \leq \varepsilon N} \overline{W}_{\varepsilon,N}^{\omega}(\eta, x, y) \right) f(\eta) \bar{\nu}_p^\rho(\, d\eta) \right].
\]

The second term in the last expression can be treated easily as in the previous proof and the first one in the same way as in the proof of Lemma 4.2 in Koukkous (1999). This concludes our proof.

4 Proof of Theorem 2.2

The proof of lower bound presented in Benois & al. (1995) is easily adapted in this case using some computations already developed in the previous proof.
of super-exponential estimate and some arguments presented in the below upper bound’s proof. We therefore omit details for the reader.

Let \( H \in \mathcal{C}_{K}^{1,2}([0, T] \times \mathbb{R}) \) and \( \gamma \in \mathcal{C}_{\varphi}(\mathbb{R}) \). From Girsanov’s formula, the Radon-Nikodym derivative of \( \mathbb{P}^{p,H}_{\gamma,N} \) with respect to \( \mathbb{P}^{N}_{\rho,p} \) is given by

\[
\exp N\left\{ \tilde{J}_{H}^{1}(\pi_{t}^{N}) + h^{p,N}_{\gamma}(\pi_{0}^{N} | \rho) - N \int_{0}^{T} \sum_{x,y} p_{x} g(\eta_{s}(x)) T(y) \left[ e^{(H(t, \frac{x+y}{N}) - H(t, \frac{x}{N})]} - 1 \right] \, ds \right\}
\]

where \( h^{p,N}_{\gamma}(\cdot | \rho) : \mathcal{M}_{+} \rightarrow \mathbb{R} \) is defined by

\[
h^{p,N}_{\gamma}(\mu | \rho) = \left< \mu, \log \left( \frac{\Phi(\gamma(\cdot))}{\Phi(\rho)} \right) \right> - \frac{1}{N} \sum_{x} \log \left[ \frac{Z(\Phi(\gamma(x/N))p_{x}^{-1})}{Z(\Phi(\rho)p_{x}^{-1})} \right]
\]

and

\[
\tilde{J}_{H}^{1}(\mu) = \left< \mu_{T}, H_{T} \right> - \left< \mu_{0}, H_{0} \right> - \int_{0}^{T} \left< \mu_{t}, \partial_{t} H_{t} \right> \, dt.
\]

**Upper bound:**

The proof is written only for fixed compact subsets \( C \) of \( \mathcal{D}([0, T], \mathcal{M}_{+}) \). To extend it to closed subsets, one needs exponential tightness for \( Q^{N}_{\rho,p} \). It can be easily obtained via the proof of Benois (1996) (see also Lemma V.1.5 in Kipnis-Landim (1999)).

For every \( q > 1 \),

\[
Q^{N}_{\rho,p}(C) = \mathbb{E}^{N}_{p,p} \left[ \left( \frac{d\mathbb{P}^{N}_{\rho,p}}{d\mathbb{P}^{p,H}_{\gamma,N}} \right)^{1/q} \left( \frac{d\mathbb{P}^{p,H}_{\gamma,N}}{d\mathbb{P}^{N}_{\rho,p}} \right)^{1/q} 1_{\{\pi_{t}^{N} \in C\}} \right].
\]

Let \( \vartheta_{\varepsilon} \) be the approximation of identity defined by \( (2\varepsilon)^{-1} 1_{[-\varepsilon, \varepsilon]}(x) \) and \( * \) the classic convolution product.

For \( 0 \leq s \leq T \), let

\[
u^{p,H}_{\vartheta_{\varepsilon},N}(\eta_{s}) = \frac{\sigma}{2N} \sum_{k} \{ \partial_{x}^{2} H(s, k/N) + [\partial_{x} H(s, k/N)]^{2} \} \{ p_{k} g(\eta_{s}(k)) - \Phi(\eta_{s}^{N}(k)) \}
\]

and

\[
u^{p,N,H}_{\vartheta_{\varepsilon},N}(\eta_{s}) = \frac{1}{N} \sum_{k} p_{k} g(\eta_{s}(k)) \left\{ \sum_{j} T(j)N^{2} \left[ e^{(H(t, \frac{k+j}{N}) - H(t, \frac{k}{N})]} - 1 \right] \right\}
\]
\[-\frac{\sigma}{2} \left\{ \partial_x^2 H(s, k/N) + (\partial_x H(s, k/N))^2 \right\} \].

From (17), a simple computation shows that we can rewrite \(\frac{dP_{\rho,p}}{dP_{\gamma,N}}\) as

\[
\exp N \left\{ -\tilde{J}_H^1(\pi_T^N) + J_H^2(\pi^N \ast \vartheta_{\varepsilon}) - h_{\gamma}^{p,N}(\pi_0^{N}|\rho) + \int_0^T \left\{ u_{\varepsilon,N}(\eta_s) + u_{N,H}(\eta_s) \right\} ds \right\}
\]

Thus, \(\frac{1}{N} \log Q_{\rho,p}^N(\mathcal{C})\) is bounded above by

\[
\frac{1}{q} \sup_{\pi \in \mathcal{C}} \left\{ -J_H^1(\pi_T) + J_H^2(\pi \ast \vartheta_{\varepsilon}) - h_{\gamma}^{p,N}(\pi_0|\rho) \right\}
\]

\[
+ \frac{1}{N} \log \mathbb{E}_{\rho,p}^N \left[ \left( \frac{dP_{\rho,p}}{dP_{\gamma,N}} \right)^{1/q} \exp \left\{ \frac{N}{q} \int_0^T \left( u_{\varepsilon,N}(\eta_s) + u_{N,H}(\eta_s) \right) ds \right\} \right]
\]

Let \(\tilde{H}\) be a real continuous function with the same support as \(\sup_t |H_t|\), such that it bounds above \(\sup_{0 \leq t \leq T} [\partial_x^2 H_t] + (\partial_x H_t)^2 + |H_t|\).

Let \(C_0 \in \mathbb{N}\) such that \(supp H \subset [0, T] \times [-(C_0 - 1), (C_0 - 1)]\). Using Hölder’s inequality, we show that, for \(q' \in \mathbb{R}\) such that \((1/q) + (1/q') = 1\), the second term in (18) is bounded above by

\[
\frac{1}{3Nq'} \log \mathbb{E}_{\rho,p}^N \left[ \exp \left\{ \frac{3Nq'}{q} \left( \int_0^T u_{\varepsilon,N}(\eta_s) ds - \int_0^T \frac{\alpha}{N} \sum_k \tilde{H}(\frac{k}{N}) \omega(\eta_s(k)) ds \right) \right\} \right]
\]

\[
+ \frac{1}{3Nq'} \log \mathbb{E}_{\rho,p}^N \left[ \exp \left\{ \frac{3Nq'}{q} \int_0^T u_{N,H}(\eta_s) ds \right\} \right]
\]

\[
+ \frac{1}{3Nq'} \log \mathbb{E}_{\rho,p}^N \left[ \exp \left\{ \frac{3Nq'}{q} \left( \frac{\alpha}{N} \int_0^T \sum_k \tilde{H}(\frac{k}{N}) \omega(\eta_s(k)) ds \right) \right\} \right]
\]

Using similar arguments as in the proof of Lemma 3.2 (see (12)), we check that the last term in (19) is bounded above by

\[
R_1(\alpha, q, H) = \frac{2C_0}{3q'} \log \nu_{\Phi(\rho)\alpha^{-1}} \left[ e^{\left\{ \frac{2q'q}{q} \|H\|_{\infty} \omega(\eta(0)) \right\}} \right]
\]
which vanishes as \( \alpha \downarrow 0 \) for each fixed \( q \) and \( H \) thanks to assumption \([H4]\).

From assumption \([H2]\), we check that \( g(k) \leq g^* k \) for all \( k \in \mathbb{Z} \) and therefore \( \Phi(\rho) \leq g^* \rho \). Thus, we repeat the same argument as above, a simple computation shows that the second term in (19) is bounded above by

\[
R_2(q, H, N) = \frac{2C_0}{3q'} \log \nu_{\Phi(\rho)\alpha_0^{-1}} \left[ e^{\frac{\beta}{2} \nu(0)} \right]
\]

where \( \beta = \beta(T, g^*, H, a_1, q, \sigma) \).

For each fixed \( q \) and \( H \), it is easy to see that \( R_2(q, H, N) \) vanishes as \( N \uparrow \infty \).

Let us turn to the first term in (19) and denote \( R_3(\alpha, q, H, \varepsilon) \) its limit when \( N \uparrow \infty \). From lemma 3.2 we check easily that

\[
\lim_{\varepsilon \to 0} R_3(\alpha, q, H, \varepsilon) = 0
\]

for all \( \alpha > 0 \), \( q > 1 \) and smooth function \( H \).

In the other hand notice that by a simple computation and from the ergodicity and stationarity of \( m \), we prove that \( h_{p, N}^\gamma(\pi_0|\rho) \) converges (uniformly in \( \pi \in \mathcal{C} \)) to \( h_{\gamma}(\pi_0|\rho) \) when \( N \uparrow \infty \) where

\[
h_{\gamma}(\mu|\rho) = \left\langle \mu, \log \left( \frac{\Phi(\gamma(\cdot))}{\Phi(\rho)} \right) \right\rangle - \int_{\mathbb{R}} \mathbb{E}_m \left[ \log \left( \frac{Z(\Phi(\gamma(x)p_0^{-1}))}{Z(\Phi(\rho)p_0^{-1})} \right) \right] \, dx.
\]

Moreover, we check also that

\[
\inf_{\gamma} h_{\gamma}(\pi_0|\rho) = h(u_0|\rho).
\]

We therefore proved that \( \lim_{N \to \infty} (1/N) \log Q_{p,p}^N(\mathcal{C}) \) is bounded above by

\[
\inf_{H,\gamma,q,\alpha,\varepsilon} \left\{ \frac{1}{q} \sup_{\pi \in \mathcal{C}} \left\{ -J_H^1(\pi) + J_H^2(\pi^* \theta_{\varepsilon}) - h_{\gamma}(\pi_0|\rho) \right\} + R_3(\alpha, q, H, \varepsilon) + R_1(\alpha, q, H) \right\}
\]

where the infimum is taken over all \( H \in \mathcal{C}_K^{1,2}([0, T] \times \mathbb{R}) \), \( \gamma \in \mathcal{C}_p(\mathbb{R}) \), \( q > 1 \), \( \alpha > 0 \) and \( \varepsilon > 0 \).
At this level, using the continuity of $J^2_H(\cdot \ast \vartheta_\varepsilon)$ for every fixed $H$ and $\varepsilon > 0$, the compacity of $C$ and the arguments developed in (Kipnis & al. (1989)) to permute the supremum and infimum, we check that this last expression is bounded above by

$$- \inf_{\pi \in C} \sup_{H,\gamma, q, \alpha, \varepsilon} \left\{ \frac{1}{q} \left\{ -J^1_H(\pi) + J^2_H(\pi \ast \vartheta_\varepsilon) - h_\gamma(\pi_0 | \rho) \right\} + R_3(\alpha, q, H, \varepsilon) + R_1(\alpha, q, H) \right\}$$

We conclude therefore the proof by letting $\varepsilon \downarrow 0$, $\alpha \downarrow 0$ and $q \downarrow 1$.

Acknowledgements. A.K. thanks Fapesp support (FAPESP n. 99/06918-0). This work is part of FAPESP Tematico n. 95/0790-1, and FINEP Pronex n. 41.96.0923.00.

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