A constructive approach to Euler hydrodynamics for attractive processes. Application to $k$-step exclusion

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Abstract

We derive by a constructive method the hydrodynamic behavior of attractive processes with irreducible jumps and product invariant measures. Our approach relies on (i) explicit construction of Riemann solutions without assuming convexity, which may lead to contact discontinuities and (ii) a general result which proves that the hydrodynamic limit for Riemann initial profiles implies the same for general initial profiles. The $k$-step exclusion process provides a simple example. We also give a law of large numbers for the tagged particle in a nearest neighbor asymmetric $k$-step exclusion process. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Among the most studied conservative particle systems are the simple exclusion and the zero-range processes. They are attractive processes, and possess a one-parameter...
family of product extremal invariant and translation invariant measures, that we denote by \( \{ \nu_x \}_{x} \), where \( \nu \) represents the mean density of particles per site: for simple exclusion \( \nu \in [0,1] \), and for zero-range \( \nu \in [0,\infty) \) (see Liggett, 1985, Chapter VIII; Andjel, 1982). Both belong to a more general class of systems with similar properties, called misanthrope processes (see Cocozza-Thivent, 1985).

Their hydrodynamic behavior in the asymmetric case in dimension 1 was derived in a constructive way in Andjel and Vares (1987), under Riemann initial condition, i.e. the initial product measure with densities \( \lambda \) to the left of the origin and \( \rho \) to its right, denoted by \( \mu_{\lambda,\rho} \). In the hydrodynamic equation

\[
\begin{align*}
\frac{\partial u}{\partial t} + \frac{\partial G(u)}{\partial x} &= 0, \\
u(x,0) &= u^0(x) = \lambda 1_{\{x<0\}} + \rho 1_{\{x\geq0\}},
\end{align*}
\]

\( u(x,t) \) is the density at the macroscopic site \( x \) and time \( t \), and \( G \) the mean flux of particles through the origin. Their method relies on attractivity, existence of product invariant measures, and concavity of \( G \). The latter is true for the exclusion process, but generally not for zero-range or misanthrope processes. Another constructive method was introduced by Seppäläinen (see e.g. Seppäläinen, 1998) for the totally asymmetric nearest neighbor exclusion process with arbitrary initial profiles. Its key tool is a microscopic Lax–Hopf formula, which applies to some other systems, but requires flux concavity or convexity. A nonconstructive proof was given by Rezakhanlou (1991) for general misanthrope processes and arbitrary initial profiles. His argument is based on the connection between attractivity and Kružkov entropy inequalities (see Kružkov, 1970). Landim (1993) proved that hydrodynamic limit for attractive systems with product invariant measures implied conservation of local equilibrium.

In this paper, we present a constructive approach to hydrodynamics of attractive systems, without convexity assumptions, assuming existence of product invariant measures. However the latter assumption is not so essential. In a forthcoming paper, we shall demonstrate how it can be relaxed by a slight extension of our arguments.

First, in Section 2, we show that the constructive argument of Andjel and Vares (1987) can be generalized without any convexity condition. In this case Eq. (1) (called the scalar Buckley–Leverett equation, in for instance Holden and Risebro (1991) or Holden (1997)), is no longer “a genuinely nonlinear conservation law”, using the language of Lax (1973). Such a solution can develop entropy shocks, rarefaction fans as well as contact discontinuities. The example that motivated us was the \( k \)-step exclusion process, a conservative approximation of the long range exclusion process (Liggett, 1980), introduced in Guiol (1999). We give an explicit construction of entropy solutions to (1), which extends the one in Ballou (1970). We then use this extension to deduce conservation of local equilibrium in the Riemann case.

Next, in Section 3, we show that for a general class of models (including all the models considered in this paper) the hydrodynamic limit for Riemann initial profiles is actually sufficient to imply that for arbitrary initial profiles. This follows from the fact that Riemann solutions characterize other entropy solutions, a property related to Glimm’s scheme (for the latter, see e.g. Serre, 1996). In Section 4 we review various
examples, among which $k$-step exclusion, and discuss nonconvexity. Finally, in Section 5, we prove a law of large numbers for a tagged particle in a nearest neighbor $k$-step exclusion process.

2. Local equilibrium for the Riemann problem

We consider particle systems on $\mathbb{Z}$ with state space $X := \mathbb{N}\mathbb{Z}$ or $X := \{0, \ldots, N\}^\mathbb{Z}$ for some exclusion constant $N$. We denote the space shift on $X$ by $\tau$ and the partial product order on $X$ by $\leq$. We will work with a conservative, translation invariant, finite range, attractive Feller process on $X$, $(\eta_t)_{t \geq 0}$ with generator $L$ and semigroup $S(\cdot)$. By attractive we mean (among other equivalent definitions) that the stochastic order on $X$ is preserved by $S(\cdot)$. See Section 4 for more details on the form of $L$, which are not required here.

Let $\mathcal{I}$ (resp. $\mathcal{S}$) denote the set of invariant (resp. translation invariant) measures of the process. We suppose that $\mathcal{I} \cap \mathcal{S}$ is a one-parameter family of product measures. The main result of this section is the conservation of local equilibrium when the family of initial measures is determined by a step function profile $u^0$. We show that the density profile at time $t$ is the entropy solution of

$$\begin{cases}
\partial_t u + \partial_x G(u) = 0, \\
u(x, 0) = u^0(x).
\end{cases}$$

The macroscopic flux $G$ is obtained as follows. First we define the microscopic current $j(\eta)$ by

$$j(\eta) = L \left[ \sum_{i > 0} \eta(i) \right].$$

Note that this is only a formal computation since $\sum_{i > 0} \eta(i)$ is not a local function; however, because the generator $L$ involves only local interactions, we do obtain a well-defined local function. Next, we set $G(u) = \nu_u[j(\eta)]$. We shall assume that $G$ thus defined is a $C^2$ function. This assumption is not necessary, but simplifies the proof of Proposition 2.1 below, and is immediately checked in all examples from Section 3. However, Proposition 2.1 and the general theory of (3) extend to Lipschitz continuous flux.

Why $G$ defined above is indeed the macroscopic flux is suggested heuristically as follows. Let $\mu_\eta$ denote the law of $\eta_t$. We take for granted that the system is in local equilibrium on Euler time scale, i.e. $\tau_{[N\lambda]}[\mu_{N\lambda}]$ converges weakly to $\nu_{u(\cdot,t)}$ as $N \to \infty$, for some limiting density profile $u(\cdot, \cdot)$. Take any $x, y \in \mathbb{R}$ with $x < y$. Then (4) implies that

$$\frac{d}{dt} \mu_{N\lambda} \left[ \sum_{N\lambda \leq i \leq N\gamma} \eta(i) \right] = \tau_{[N\lambda]}[\mu_{N\lambda}[j(\eta)] - \tau_{[N\gamma]}[\mu_{N\lambda}[j(\eta)].$$
Thus, by the local equilibrium assumption,
\[ \frac{d}{dt} \int_x^y u(z,t) \, dz = G(u(x,t)) - G(u(y,t)), \]
weakly in time. This is exactly the integrated form of (3), i.e. \( u \) must be a weak solution.

2.1. The equation in the Riemann case

We here give a brief summary of results concerning the solution of the Riemann problem for a scalar conservation law with a continuously differentiable flux \( G \), based on the papers by Ballou (1970), Conway and Smoller (1966), and the book by Godlewski and Raviart (1991). This will motivate the formulation of the theorem as well as some aspects of the proof.

Existence of weak solutions to the Cauchy problem given by Eq. (3) with initial condition of bounded variation was proved in Conway and Smoller (1966) under the assumption that \( G \in C^1(\mathbb{R}) \). In order to obtain uniqueness, we require furthermore our solutions to satisfy the

**Condition E** (Olešnik). Let \( x(t) \) be any curve of discontinuity of the weak solution \( u(x,t) \), and let \( v \) be any number lying between \( u^- := u(x(t) - 0, t) \) and \( u^+ := u(x(t) + 0, t) \). Then except possibly for a finite number of \( t \),
\[ S[v; u^-] \geq S[u^+; u^-], \]
where
\[ S[v; w] := \frac{G(w) - G(v)}{w - v}. \]

The geometric interpretation is that the chord connecting \( u^- \) and \( u^+ \) on the graph of \( G \) lies below (resp. above) the graph if \( u^- < u^+ \) (resp. \( u^- > u^+ \)). The following two conditions are necessary and sufficient for a piecewise smooth function \( u(x,t) \) to be a weak solution to Eq. (3) (see Ballou, 1970):

1. \( u(x,t) \) solves Eq. (3) at points of smoothness.
2. If \( x(t) \) is a curve of discontinuity of the solution then the Rankine–Hugoniot condition \( \frac{d(x(t))}{dt} = S[u^+; u^-] \) holds along \( x(t) \).

Moreover Condition E is sufficient to ensure the uniqueness of piecewise smooth weak solutions to Eq. (3).

For the initial value problem (1), both the equation and the initial condition are invariant under the scaling \( x \to Ax, \ t \to At \). Therefore we can write the (self-similar) weak solution to (1) as
\[ u(x,t) = u(x/t, 1) \equiv u(v, 1), \quad v = x/t. \]

To describe the solution to (1) for a flux function \( G \in C^2(\mathbb{R}) \), we proceed in three steps: a strictly convex (or concave) flux, a flux with one inflexion point, finally the general case.
Step 1. We assume first that \( G \) is strictly convex (concave), and exhibit the only two possible types of solutions.

If \( \lambda > \rho \ (\rho > \lambda) \), the characteristics starting from \( x \leq 0 \) have a speed (given by \( H := G' \)) greater than the speed of those starting from \( x > 0 \). If the characteristics intersect along a curve \( x(t) \), then Rankine–Hugoniot condition will be satisfied if

\[
x'(t) = S[u^+; u^-] = \frac{G(\lambda) - G(\rho)}{\lambda - \rho} = S[\lambda; \rho].
\]

Thus

\[ u(x,t) = \begin{cases} 
\lambda, & x \leq S[\lambda; \rho]t, \\
\rho, & x > S[\lambda; \rho]t
\end{cases} \]  

(5)

is a weak solution. The convexity of \( G \) implies that Condition E is satisfied across \( x(t) \). Therefore (5) is the unique entropic solution in this case, referred to as a shock in the sense of Lax (1973).

If \( \lambda < \rho \ (\rho < \lambda) \), then the characteristics starting, respectively, from \( x \leq 0 \) and \( x > 0 \) never meet. Moreover they never enter the space–time wedge between lines \( x = H(\lambda)t \) and \( x = H(\rho)t \). We can choose values in this region to obtain the so-called continuous solution with a rarefaction fan:

\[ u(x,t) = \begin{cases} 
\lambda, & x \leq H(\lambda)t, \\
h(x/t), & H(\lambda)t < x < H(\rho)t, \\
\rho, & H(\rho)t < x,
\end{cases} \]

(6)

where \( h \) is the inverse of \( H \). It is possible to define piecewise smooth weak solutions with a jump occurring in the wedge satisfying the Rankine–Hugoniot condition. But the convexity of \( G \) prevents such solutions from satisfying Condition E. Thus (6) is the unique entropic solution in this case.

Step 2. We suppose that \( G \) has a single inflexion point \( a \in (0,1) \) and \( G(u) \) is strictly convex in \( 0 \leq u < a \) and strictly concave in \( a < u \leq 1 \). There, in addition to the shock and the continuous solution with a rarefaction fan we have a third type of solution, with a jump discontinuity, which does not involve intersection of characteristics. We rely on Ballou (1970). The quotation in brackets is the original number of lemmas, propositions, ... in Ballou (1970).

**Definition 2.1 ([B def2.1]).** For any \( u < a \), define \( u^* := u^*(u) \) as

\[ u^* = \sup \{ w > u : S[u; w] > S[v; u] \ \forall v \in (u, w) \}. \]

For any \( u > a \), define \( u_* := u_*(u) \) as

\[ u_* = \inf \{ w < u : S[u; w] > S[v; u] \ \forall v \in (w, u) \}. \]

In other words, for \( u < a \), \( u^* \) is the first point where \( G^c \), the *upper convex envelope* of \( G \) on \([u, +\infty)\), coincides with \( G \), and when \( u > a \), \( u_* \) is the first point where the *lower convex envelope* \( G_c \) of \( G \) on \((-\infty, u]\) coincides with \( G \).
The lower convex envelope $G_c$ of a function $G$ in an interval $[\beta, \gamma]$ can be defined by (see Rockafellar, 1970, p. 36):

Let the epigraph of $G$ on $[\beta, \gamma]$ be the set

$$\text{epi} G = \{(x, \mu): x \in [\beta, \gamma], \mu \in \mathbb{R}, \mu \geq G(x)\},$$

$\text{conv}(\text{epi} G)$ be the convex hull of $\text{epi} G$, i.e. the intersection of all the convex sets containing $\text{epi} G$. Then for $x \in [\beta, \gamma]$,

$$G_c(x) = \inf \{\mu: (x, \mu) \in \text{conv}(\text{epi} G)\}.$$

(For the upper convex envelope reverse the inequality in the definition of the epigraph and take the supremum instead of the infimum in the last expression).

The function $G_c$ is convex with a nondecreasing derivative $H_c$.

**Lemma 2.1** ([Lemma 2.2], [Lemma 2.4]). Let $w < a$ be given, and suppose that $w^* < \infty$. Then

(a) $S[w, w^*] = H(w^*)$;
(b) $w^*$ is the only zero of $S[u, w] - H(u)$, $u > w$.

If $\lambda < \rho < a$, the relevant part of the flux function is convex and the unique entropic weak solution is the continuous solution with a rarefaction fan.

If $\rho < \lambda < \rho^*$ ($\rho < a$), then $H(w) > H(\rho)$ for $\rho < w \leq a$, and $H(w) > H(\rho^*) > H(\rho)$ for $a < w < \rho^*$ since $H$ is decreasing in this region. Thus $H(\lambda) > H(\rho)$, which implies an intersection of characteristics: The unique entropic weak solution is the shock.

Let $\rho < \rho^* < \lambda$ ($\rho < a$): Lemma 2.1 applied to $\rho$ suggests that a jump from $\rho^*$ to $\rho$ along the line $x = H(\rho^*)t$ will satisfy the Rankine–Hugoniot condition. Due to the definition of $\rho^*$, a solution with such a jump will also satisfy Condition E, therefore it would be the unique entropic weak solution. Notice that since $H(\lambda) < H(\rho^*)$, no characteristics intersect along the line of discontinuity $x = H(\rho^*)t$. Ballou calls this case a contact discontinuity. The solution is defined by

$$u(x, t) = \begin{cases} 
\lambda, & x \leq H(\lambda)t, \\
h_2(x/t), & H(\lambda)t < x \leq H(\rho^*)t, \\
\rho, & H(\rho^*)t < x,
\end{cases}$$

(7)

where $h_2$ is the inverse of $H$ restricted to $(a, +\infty)$.

Corresponding cases on the concave side of $G$ are treated similarly, with $h_1$ the inverse of $H$ restricted to $(-\infty, a)$.

**Step 3.** Both the shock and the contact discontinuity occur across straight lines whose slope ($S[\lambda; \rho]$ in (5), $S[\rho; \rho^*] = H(\rho^*)$ in (7)) equals that of a flat part of the (upper or lower) convex envelope of $G$. Away from such lines the solution is continuous and can be obtained as the solution of Eq. (1) (it was $\lambda, \rho$ or an inverse of $H$). This idea generalizes to an equation with a $C^2$ flux function for which we now describe
the solution to the Riemann problem. Let us assume that \( \lambda < \rho \), that is an increasing initial profile.

Since \( H_c \) is continuous and nondecreasing in \([\lambda, \rho]\) it has an inverse function (not necessarily continuous) \( h_c \) strictly increasing on \([v_*, v^*]\), where \( v^* = H_c(\rho), \ v_* = H_c(\lambda) \). For instance we may define

\[
h_c(v) = \sup\{ x \in [\lambda, \rho]: H_c(x) = v \},
\]
in which case we get a right-continuous function. The set of discontinuities of \( h_c \) is given by

\[
\Sigma_{low}(G) = \{ v \in [v_*, v^*]: H_c \equiv v, \text{ in a nonempty open subinterval of } [\lambda, \rho] \}.
\]

It is a set of isolated points, thus countable as well as its closure \( \hat{\Sigma}_{low}(G) \). In the appendix we prove the following properties for \( v \in [v_*, v^*] \):

**Lemma 2.2.** \( G(x) - vx \) and \( G_c(x) - vx \) have the same global minimum value on \([\lambda, \rho]\); \( h_c(v - 0) \) and \( h_c(v + 0) \) are, respectively, the smallest and greatest \( x \) where global minimum of \( G(x) - vx \) is attained, and the set of global minima of \( G_c(x) - vx \) on \([\lambda, \rho]\) is the interval \([h_c(v - 0), h_c(v + 0)]\).

**Corollary 2.1.** If \( v \in \Sigma_{low}(G) \), the graph of \( G_c \) between \( h_c(v - 0) \) and \( h_c(v + 0) \) is a chord of the graph of \( G \), with slope \( v \), lying below the graph of \( G \). If \( v \not\in \Sigma_{low}(G) \), the graphs of \( G \) and \( G_c \) coincide and are strictly convex in a neighborhood of \( h_c(v) \).

Let us extend the domain of \( h_c \) to \( \mathbb{R} \) by setting \( h_c(v) = \lambda \) if \( v < v_* \) and \( h_c(v) = \rho \) if \( v > v^* \). We then have

**Proposition 2.1.** Let \( G \in \mathcal{C}^2(\mathbb{R}) \). Then the self-similar entropy weak solution of (1) is given by \( u(v, 1) = h_c(v) \). In particular, at points of continuity, \( u(v, 1) \) is the unique global minimum of \( G(x) - vx \).

**Proof.** By Corollary 2.1, \( G \) is strictly convex in a neighborhood of \( h_c(v) \) for any \( v \not\in \Sigma_{low}(G) \), and \( h_c \) is locally the inverse of \( G' \). It follows that \( u \) is \( \mathcal{C}^1 \) outside the lines with slopes \( v \in \Sigma_{low}(G) \). In that domain the solution is locally a rarefaction fan, and thus satisfies (1) in strong sense. It remains to check that discontinuities satisfy the Rankine–Hugoniot condition and Condition E across discontinuity lines with slopes \( v \in \Sigma_{low}(G) \). But this follows immediately from Corollary 2.1, since we then have \( u(x \pm 0, t) = h_c(x/t \pm 0) \).

**Remark.** In the case when \( G \in \mathcal{C}^2(\mathbb{R}) \) is such that \( G'' \) vanishes only finitely many times, the above construction is equivalent to Ballou (1970), Theorems 2.1 and 4.1 (see also Godlewski and Raviart, 1991, Section II.6). Indeed, \( G_c \) consists of a finite succession of chords connecting two points on the graph of \( G \), lying below the latter, and of strictly convex portions of \( G \). The set \( \Sigma_{low}(G) \) is finite, and \( h_c \) has finitely many discontinuities.
2.2. Local equilibrium

We are able now to state the main result of this section.

Theorem 2.1. Let \( v \in \mathbb{R}, \ 0 \leq \lambda < \rho \ (0 \leq \rho < \lambda) \), and \( \mu_{\lambda, \rho} \) the initial product measure on \( \mathbb{Z} \) with densities \( \lambda \) for \( x \leq 0 \) and \( \rho \) for \( x > 0 \). Then

\[
\lim_{t \to \infty} \mu_{\lambda, \rho} \tau_{[\varepsilon r]} S(t) = v_{u(v, 1)}
\]

at points of continuity, i.e. \( v \in (\Sigma_{\text{low}}(G))^c \), where \( u(v, 1) \) is the solution defined in Proposition 2.1.

We will follow the scheme introduced in Andjel and Vares (1987), where the authors obtained the conservation of local equilibrium for the one-dimensional zero-range process in the Riemann case, with density profile given by Eq. (1). However they needed the additional assumption that \( G \) was concave.

We can use the parts of their proof relying on attractivity, but we have to replace the arguments based on the concavity of the flux by properties of the solution of (1).

Informally speaking, they first showed that a weak Cesàro limit of (the measure of) the process is an invariant and translation invariant measure. Then they computed the (Cesáro) limiting density inside a macroscopic box, which equals the difference of the edge values of a flux function. These propositions were based on monotonicity, and on the characterization of invariant and translation invariant measures, thus we can quote them, and take them for granted.

Lemma 2.3 ([AV 3.1]). Let \( \mu \) be a probability measure on \( \{0, 1\}^\mathbb{Z} \) such that

(a) \( \nu_\rho \leq \mu \leq \nu_\lambda \) for some \( 0 \leq \rho < \lambda \),
(b) either \( \mu \tau_1 \leq \mu \) or \( \mu \tau_\lambda \geq \mu \).

Then any sequence \( T_n \to \infty \) has a subsequence \( T_{nm} \) for which there exists a dense (countable) subset \( D \) of \( \mathbb{R} \) such that for each \( v \in D \),

\[
\lim_{m \to \infty} \frac{1}{T_{nm}} \int_0^{T_{nm}} \mu \tau_{[\varepsilon r]} S(t) \, dt = \mu_v
\]

for some \( \mu_v \in \mathcal{I} \cap \mathcal{S} \).

Lemma 2.4 ([AV 3.2]). For \( v \in D \), we can write \( \mu_v = \int v_x \gamma_v (dx) \), where \( \gamma_v \) is a probability on \( [\rho, \lambda] \). Also, if \( u < v \) are in \( D \),

\[
\lim_{m \to \infty} \mu_s(T_{nm}) \left( \frac{1}{T_{nm}} \sum_{[uT_{nm}]} \eta(x) \right) = F(v) - F(u)
\]

with, for \( w \in D \), \( F(w) = \int [wz - G(z)] \gamma_w (dz) \).

Lemma [AV 3.2] was proved originally in Andjel and Vares (1987) for the zero-range process. However, their computations rely exactly on the expression of the microscopic
current given in (4), so it carries over to more general jump rates of the form (14) discussed in Section 4 below.

The difficult part is then to prove that \( \gamma_v \) is the Dirac measure concentrated on \( u(v, 1) \). They did it in Lemma [AV 3.3] and Theorem [AV 2.10] thanks to the concavity of their flux function. At this point we will use below the minimum (maximum) principle (Proposition 2.1) that characterized the weak entropic solution of (1). Then, we prove that Cesàro limit implies the weak limit via monotonicity arguments, that include the following proposition of Andjel and Vares (1987) (also based on monotonicity).

**Proposition 2.2** ([AV 3.5]). If \( \mu \) satisfies

(a) \( \mu \leq \nu \), (b) \( \mu 1 \geq \mu \), (c) there exists \( v_0 \) finite so that

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \mu_{[\theta]}S_2(t) \, dt = v_\lambda
\]

for all \( v > v_0 \). Then

\[
\lim_{t \to \infty} \mu_{[\theta]}S_2(t) = v_\lambda \quad \text{for all } v > v_0.
\]

**Proof of Theorem 2.1.** Recall that we are assuming \( 0 \leq \lambda < \rho \). Relying on (8), we now establish the inequality

\[
\int_{\lambda}^{\rho} (G(\theta) - \nu \theta) \gamma_v(\theta) \, d\theta \leq \inf_{\theta \in [\lambda, \rho]} (G(\theta) - \nu \theta)
\]

for all \( v \notin \Sigma_{low}(H) \). Indeed, since \( v \) is a continuity point of \( u(., 1) \), \( (G(\cdot) - \nu \cdot) \) attains its global minimum in \([\lambda, \rho] \) at \( u(v, 1) \), which, combined with (9), implies that \( \gamma_v = \delta_{u(v, 1)} \).

Using that our process is monotone and finite range, we can proceed as in the beginning of the proof of Lemma [AV3.3], and get two finite values \( \bar{v} \) and \( \tilde{v} \) so that: If \( v \in D \) and \( v > \bar{v} \), then \( \gamma_v = \delta_{\bar{v}} \), while \( \gamma_v = \delta_{\tilde{v}} \) if \( v < \bar{v} \).

Choose \( u_1 \) and \( v_1 \) such that \( u_1 < \bar{v} < v < \tilde{v} < v_1 \). Then \( \gamma_{v_1} = \delta_{\rho} \) and \( \gamma_{u_1} = \delta_{\lambda} \). Since \( \mu_{\lambda, \rho} \geq v_\lambda \) attractivity implies

\[
\lim_{t \to \infty} \mu_{\lambda, \rho}S_1(t) \left( \frac{1}{t} \sum_{[u_1t]} \eta(x) \right) \geq (v - u_1)\lambda.
\]

Now from (8)

\[
\lim_{t \to \infty} \mu_{\lambda, \rho}S_1(t) \left( \frac{1}{t} \sum_{[u_1t]} \eta(x) \right) = (v_1\rho - G(\rho)) - (u_1\lambda - G(\lambda)).
\]

Combining this with (10) gives

\[
\limsup_{t \to \infty} \mu_{\lambda, \rho}S_1(t) \left( \frac{1}{t} \sum_{[v_1t]} \eta(x) \right) \leq (v_1\rho - G(\rho)) - (v\lambda - G(\lambda)).
\]
Let \( \theta \in [\lambda, \rho] \). Repeating the above arguments for \( \mu_{\theta, \rho} \) we obtain

\[
\limsup_{t \to \infty} \mu_{\theta, \rho} S(t) \left( \frac{1}{t} \sum_{[vt]} \eta(x) \right) \leq (v_1 \rho - G(\rho)) - (v \theta - G(\theta)).
\]

We therefore have, using (8) again, and \( \mu_{\lambda, \rho} \leq \mu_{\theta, \rho} \)

\[
(v_1 \rho - G(\rho)) - \int_{\lambda}^{\rho} (v \alpha - G(\alpha)) \gamma_\alpha(d\alpha)
\]

\[
\leq \limsup_{t \to \infty} \mu_{\lambda, \rho} S(t) \left( \frac{1}{t} \sum_{[vt]} \eta(x) \right)
\]

\[
\leq \limsup_{t \to \infty} \mu_{\theta, \rho} S(t) \left( \frac{1}{t} \sum_{[vt]} \eta(x) \right)
\]

\[
\leq (v_1 \rho - G(\rho)) - (v \theta - G(\theta)).
\]

Putting this all together yields (9), thus the Cesáro limit of \( \mu_{\lambda, \rho} \tau_{[vt]} S(t) \) is \( v_{\theta(1)} \). In order to prove conservation of local equilibrium we want to replace the Cesáro limit by convergence in distribution (\textit{weak} limit), that is

\[
\limsup_{t \to \infty} \mu_{\lambda, \rho} S(t) \left( \frac{1}{t} \sum_{[vt]} \eta(x) \right) \]

\[
\leq \limsup_{t \to \infty} \mu_{\theta, \rho} S(t) \left( \frac{1}{t} \sum_{[vt]} \eta(x) \right)
\]

\[
\leq (v_1 \rho - G(\rho)) - (v \theta - G(\theta)).
\]

For \( v < v_* \) and \( v > v^* \), that result follows from Proposition 2.2. Using that \( u(\nu, 1) \) is a nondecreasing function, attractivity, that \( \tau_1 \mu_{\lambda, \rho} \geq \mu_{\lambda, \rho} \), to derive (11) for all \( v \) it is sufficient to prove it for a dense subset of \( v \)'s in \([v_*, v^*] \): Let \( v \) be an interior point of \([\Sigma_{\text{low}}(G)]\), let \( \tilde{\mu}_v \) be a subsequential weak limit of \( \mu_{\lambda, \rho} \tau_{[vt]} S(t) \). There is a unique \( \theta \in (\lambda, \rho) \) such that \( H_c(\theta) = v \), and \( H(\theta) = v \) (without necessarily being unique). There exists a neighborhood \( \Theta \) of \( \theta \) on which \( G \) is strictly convex. For \( \theta' \in \Theta \), the lower convex hull of the restriction of \( G \) to \([\lambda, \theta']([\theta', \rho]) \) is the restriction of the lower convex envelope \( G_c \) to \([\lambda, \theta']([\theta', \rho]) \), and if \( \theta' < \theta \), \( v = H(\theta) > H(\theta') \). This implies

\[
v_\theta = \lim_{t \to \infty} \mu_{\lambda, \theta} \tau_{[vt]} S(t) \leq \liminf_{t \to \infty} \mu_{\lambda, \rho} \tau_{[vt]} S(t) \leq \tilde{\mu}_v
\]

where the first equality comes from Proposition 2.2. Now let \( \theta' \to \theta \), by continuity we have

\[
\tilde{\mu}_v \geq v_\theta = v_{\theta(1)}.
\]

We obtain the reverse inequality by considering \( \mu_{\theta', \rho} \). This completes the proof of the theorem when the initial profile is an increasing step function. For an initial decreasing step function we proceed in the same way using the upper (instead of the lower) convex envelope. \( \square \)

3. From Riemann to general initial profiles

In this section, we provide a model-independent argument to establish the following principle for a wide class of conservative systems with local interactions, including the
examples of this paper: if one can prove that for Riemann initial profiles (i.e. constant profiles or single steps) the hydrodynamic limit is given by the entropy solution to (3), then this remains true for any bounded measurable initial profile. This is the object of Theorem 3.2. Our idea is inspired by the Glimm’s scheme for hyperbolic systems of conservation laws (see e.g. Chapter 5 of Serre, 1996). According to the latter, it is possible to reconstruct any entropy solution by approximations using only Riemann solutions. Theorem 3.2 is a translation to particle (microscopic) level of Theorem 3.1, a p.d.e. (macroscopic) result motivated by Glimm’s scheme.

We first set some necessary definitions. \( N \in \mathbb{N}^* \) is the scaling parameter for the hydrodynamic limit, i.e. the inverse of the macroscopic distance between two consecutive sites. The empirical measure of a configuration \( \eta \) viewed on scale \( N \) is given by

\[
\nu_N(\eta, dx) = N^{-1} \sum_{x \in \mathbb{Z}} \eta(x) \delta_{x/N}(dx) \in \mathcal{M},
\]

where \( \mathcal{M} \) denotes the set of positive, locally finite measures on \( \mathbb{R} \). \( \mathcal{M} \) is equipped with the topology of vague convergence, defined by convergence for continuous test functions with compact support. Let \( u(\cdot) \) be a deterministic bounded Borel function on \( \mathbb{R} \). A sequence of random configurations \( \eta^N \) is said to have density profile \( u(\cdot) \) (resp. density profile \( u(\cdot) \) on the interval \( I \subset \mathbb{R} \)) if \( \nu_N(\eta^N, dx) \) (resp. its restriction to \( I \)) converges in probability to \( u(\cdot) \) as \( N \rightarrow \infty \); bounded support (resp. bounded mass), if there exists \( a > 0 \) independent of \( N \), such that \( P\{\eta^N(x) = 0 \text{ for every } x \notin [-Na, Na]\} \) (resp. \( P\{\sum_{x \in \mathbb{Z}} \eta^N(x) \leq Na\}) \) converges to 1 as \( N \rightarrow \infty \).

Now let \( u(\cdot, \cdot) \) be a bounded Borel function on \( \mathbb{R} \times \mathbb{R}^+ \). A sequence of processes with initial configurations \( \eta^N \) is said to have hydrodynamic limit \( u(\cdot, \cdot) \) (under Euler time scaling), if for every \( t > 0 \), the sequence \( \eta^N_{Nt} \) has density profile \( u(\cdot, t) \) (for technical simplicity we use this definition, but a stronger notion can be defined on process level). Here and in the sequel, \( \eta^N_{Nt} \) denotes the state at time \( t \) of the system starting from \( \eta^N \).

In the previous section for Riemann profiles we proved (Theorem 2.1) that the distribution of the scaled process measure \( \mu^N_{Nt} \) converges in distribution to \( u(\cdot, t)dx \). It implies that the sequence of processes has hydrodynamic limit given by \( u(x,t) \), the entropy solution to (1) (see e.g. Chapters 1 and 3, Proposition 0.4 of Kipnis and Landim, 1999).

The following result is inspired by Glimm’s scheme. Denote by \( \mathcal{B}_+ \) the set of bounded Borel functions on \( \mathbb{R} \) with values in the domain of definition of the flux function \( G \), i.e. \( \mathbb{R}^+ \) or \([0, \mathcal{N}]\). Let \( (T_t, t \geq 0) \) be an evolution semigroup on \( \mathcal{B}_+ \): for some initial density profile \( u(\cdot, 0) \), \( T_t u(\cdot) \) is the profile at time \( t \). Assume we know that for Riemann profiles \( u, T_t u(\cdot) \) is the corresponding entropy solution at time \( t \). Provided \( T_t \) has some stability properties in common with entropy solutions, we can prove that \( T_t u(\cdot) \) is the entropy solution for any starting profile \( u(\cdot) \). To make the statement precise, we define

\[
F_{u(\cdot)}(x) = \int_{-\infty}^{x} u(y) \, dy, \quad A(u(\cdot), u'(\cdot)) = \sup_{x \in \mathbb{R}} \left| F_{u(\cdot)}(x) - F_{u'(\cdot)}(x) \right|
\]

for integrable density profiles \( u(\cdot), u'(\cdot) \in \mathcal{B}_+ \).
Theorem 3.1. Assume the semigroup $T$ has the following properties:

(1) For any Riemann profile $u_0(\cdot)$, $(t,x) \mapsto T_t u_0(x)$ is the entropy solution to (3) with Cauchy datum $u_0(\cdot)$.

(2) (Finite speed of discrepancies). There is a constant $v$ such that, for any $[x; y] \subset \mathbb{R}$, any two profiles $u_0(\cdot), u_0'(\cdot)$ coinciding a.e. on $[x; y]$, and any $t < (y - x)/2v$, $T_t u_0(\cdot)$ and $T_t u_0'(\cdot)$ coincide a.e. on $[x + vt; y - vt]$.

(3) (Time continuity). For every $u_0(\cdot) \in \mathcal{B}_+$ with bounded support and $t > 0$, $\lim_{\varepsilon \to 0^+} \Delta(T_t u_0(\cdot), T_{t+\varepsilon} u_0(\cdot)) = 0$.

(4) (Stability). For any $u_0(\cdot), u_0'(\cdot) \in \mathcal{B}_+$ with bounded support and $t > 0$, $\Delta(T_t u_0(\cdot), T_t u_0'(\cdot)) \leq \Delta(u_0(\cdot), u_0'(\cdot))$.

Then, for any $u_0 \in \mathcal{B}_+$, $(t,x) \mapsto T_t u_0(x)$ is the entropy solution to (3) with Cauchy datum $u_0$.

Note that the Riemann solution with Cauchy datum 0 is 0; hence the first two assumptions imply that, if $u_0(\cdot)$ has bounded support, $T_t u_0(\cdot)$ has bounded support expanding with maximum speed $v$. It is known that the entropy solution defines an evolution semigroup on $\mathcal{B}_+$ which satisfies the last three assumptions (see Chapter 2 of Serre (1996) for assumption 2, and assumption 3 in stronger $L^1$ topology; Lax (1957) for assumption 4). This semigroup will be denoted by $U = (U_t, \ t \geq 0)$. In this case, the constant $v$ in assumption 2 is $\|G'\|_{\infty}$. Therefore the above result says that these properties characterize all entropy solutions if we only know Riemann solutions.

The idea of Glimm’s scheme is to approximate an arbitrary profile with a piecewise constant profile. The key observation is that, for such a profile, successive steps will not interact on a time scale of the same order as the shortest step length. This is an immediate consequence of assumption 2. Therefore on this scale, the entropy solution can be obtained as a superposition of noninteracting Riemann waves. When interaction occurs, we replace the current profile with an approximating piecewise constant profile, and so forth.

It is natural to expect that, if a particle system has properties similar to assumptions of Theorem 3.1, hydrodynamics of form (3) for Riemann datum should imply hydrodynamics for arbitrary datum. Such properties on particle level involve a coupling of two systems. Assumption 2 means that the evolution propagates discrepancies with maximum speed $v$. This property has a microscopic analogue for conservative systems with local interactions. Namely:

Lemma 3.1. For every $x, y \in \mathbb{Z}$ with $x < y$, there exist Poisson processes $(X^x_t, \ t \geq 0)$ and $(Y^y_t, \ t \geq 0)$ with initial value 0, such that: for any two initial configurations $\eta$ and $\zeta$ coinciding on the interval $[x; y]$ and every $t > 0$, $\eta_t$ and $\zeta_t$ coincide on the interval $[x + VX_t^x, y - VY_t^y]$. The constant $V$ depends only on the generator $L$ of the dynamics and not on $x, y$ or $\eta, \zeta$.

Assumption 3 means that the macroscopic profile is not much changed in small macroscopic time, and also holds on microscopic level. Before stating the corresponding result, we need some more notations. We consider particle configurations $\eta$ with finitely...
many particles. For such configurations, we define the analogues of $F$ and $A$ above, with the same notations for simplicity:

$$F_{\eta}(x) = \sum_{y \leq x} \eta(y), \quad A(\eta, \zeta) = \sup_{x \in \mathbb{Z}} |F_{\eta}(x) - F_{\zeta}(x)|.$$ 

Denote by $A^N(\eta, \zeta) = N^{-1} A(\eta, \zeta)$ the rescaled version of $A$.

**Lemma 3.2.** Assume $\eta^N$ has bounded support. Let $\delta > 0$ and $T > 0$ be given. Then, for small enough $\varepsilon > 0$,

$$\lim_{N \to \infty} P \left( \sup_{t \in [0; T]} A^N(\eta^N_{Nt}, \eta^N_{N(t+\varepsilon)}) > \delta \right) = 0.$$ 

Lemmas 3.1 and 3.2 are proved in the appendix. To avoid technical assumptions, proofs are given for bounded jump rates. They rely on locality of interactions, but not on attractivity. Now we state a condition for the particle system that is the microscopic analogue of assumption 4. In the sequel, $o_N(1)$ will denote various sequences of random variables converging to 0 in probability.

**Definition 3.1.** We say the system is *macroscopically stable* if

$$A^N(\eta^N_{Nt}, \zeta^N_{Nt}) \leq A^N(\eta^N, \zeta^N) + o_N(1)$$

for every $t > 0$ and sequences $\eta^N$ and $\zeta^N$ of initial configurations with bounded mass.

A consequence of this condition is that the hydrodynamic limit depends only on the density profile at time 0, and not on the underlying microscopic structure (e.g. nonequilibrium product measure). More precisely, combining Lemma 3.1 and macroscopic stability, we obtain the following result, whose proof is left to the reader:

**Lemma 3.3.** Assume that (i) $\eta^N$ and $\zeta^N$ have some density profile $u_0(.)$ on $[x, y] \subset \mathbb{R}$ (ii) $\eta^N_{Nt}$ has density profile $u(.)$ on $[x + Vt, y - Vt]$, with the constant $V$ of Lemma 3.1. Then $\zeta^N_{Nt}$ has density profile $u(.)$ on $[x + Vt, y - Vt]$.

Macroscopic stability is satisfied by many particle systems, which include all those for which (3) is currently known. For nearest-neighbor attractive systems, and some nonattractive systems, it is easy to show the stronger property that $A$ is pathwise nondecreasing. See Bahadoran (2001) for a discussion and examples. For the exclusion process with irreducible jumps, the proof is given in Bramson and Mountford (2001). Their arguments rely on attractivity and an irreducibility property which are satisfied by the systems reviewed in Section 4, and therefore extend to these processes. We can now state the main result of this section.

**Theorem 3.2.** Assume a particle system is macroscopically stable, and the hydrodynamic limit under Euler time scaling for Riemann initial profiles is given by the entropy solution to (3). Then the hydrodynamic limit is given by the entropy solution for any bounded, measurable initial density profile.
Using Theorems 2.1 and 3.2, we get

**Theorem 3.3.** Assume a (conservative, translation invariant) particle system is attractive and satisfies assumption (2). Then, for every sequence of initial random configurations with bounded measurable density profile $u_0(.)$, the hydrodynamic limit under Euler time scaling is given by the entropy solution to (3) with initial datum $u_0(.)$.

Using this theorem and a result of Landim (1993) (see Kipnis and Landim, 1999, Chapter 8) it is easy to deduce conservation of local equilibrium for product initial measures with bounded measurable initial profiles. We state this result below as Corollary 3.1.

**Corollary 3.1.** Assume a particle system is attractive and satisfies (2). Let $(\mu^N)$ be a sequence of product initial measures such that

$$\mu^N(\eta(x) \in .) = v_{\mu^N,u}(\eta(x) \in .)$$

for some family $(u^{(N,x)}, N \in \mathbb{N}^*, x \in \mathbb{Z})$ satisfying

$$\lim_{N \to +\infty} \int_K |u^{(N,[Nx])} - u^0(x)| = 0$$

for any bounded $K \subset \mathbb{R}$, where $u^0$ is a bounded measurable function. Then

$$\lim_{N \to +\infty} \tau_{[Nx]} \mu^N S(Nt) = v_{\mu,u(t)}$$

for all $t > 0$, at every continuity point $x$ of $u(x,t)$, where $u(x,t)$ is the entropy solution to (3), with initial condition $u(x,0) = u^0(x)$.

We now turn to the proofs of Theorems 3.1 and 3.2. The latter is essentially a copy of the former on particle level. Our main ingredients are: (i) solutions obtained as a superposition of noninteracting Riemann waves, and (ii) a sharp approximation result to replace a general profile by a piecewise constant one. We denote by $R_{\lambda,\rho}(t,x)$ the entropy solution to the Riemann problem with left (right) density $\lambda(\rho)$.

**Lemma 3.4.** Under the hypotheses and notation of Theorem 3.1, set $v' := \max(v, \|G'\|_\infty)$. Let $x_0 = -\infty < x_1 \cdots x_n < x_{n+1} = +\infty$, and $\varepsilon := \min_k (x_{k+1} - x_k)$. Denote by $u_0(.)$ the piecewise constant profile with value $r_k$ on $I_k := (x_k, x_{k+1})$ for $k \in \{0, \ldots, n\}$. Then, for $t < \varepsilon/(2v')$, $T_tu_0(.) = U_tu_0(.) := u(.,t)$ is given by

$$u(x,t) = R_{r_{k-1},r_k}(t - x_k), \quad \forall x \in (x_{k-1} + Vt, x_k + Vt),$$

in particular,

$$u(x,t) = r_k, \quad \forall x \in (x_k + Vt, x_{k+1} - Vt).$$

The interval $(x_k + Vt, x_{k+1} - Vt)$ is a “no interaction zone” that separates two consecutive Riemann waves originating from $x_k$ and $x_{k+1}$. The proof follows immediately.
from comparing $u_0(\cdot)$ with each one of its single steps and using assumption 2 of Theorem 3.1. Using Lemma 3.3, we obtain the analogue of Lemma 3.4 for the particle system.

**Lemma 3.5.** Set $V' := \max(V, v, \|G'\|_\infty)$, with $V$ the constant in Lemma 3.1. Assume the sequence $\eta^N$ has density profile $u_0(\cdot)$ defined in Lemma 3.4. Then at times $t < \varepsilon/(2V')$, the sequence $\eta^N_N$ has density profile $U_t u_0(\cdot) = u(\cdot, t)$.

We now prove that one may replace a rather general profile by a piecewise constant profile with step lengths at least $\varepsilon$, such that the approximation is sharper than $\varepsilon$. We need a somewhat different approximation procedure than in the usual Glimm’s scheme.

**Lemma 3.6.** Assume $u(\cdot)$ has compact support and finite variation, and let $\delta > 0$. Then, for $\varepsilon > 0$ small enough, there exists an approximation $u^{\varepsilon, \delta}$ of $u$ with the following properties: $u^{\varepsilon, \delta}$ is a piecewise constant function with compact support, step lengths at least $\varepsilon$, and $A(u^{\varepsilon, \delta}, u) \leq \varepsilon \delta$.

**Proof.** Since $u$ has finite variation, it is continuous except on an at most countable set of points with first kind discontinuities; and there are finitely many points $x_1 < \cdots < x_n$ where the jump exceeds $\delta/2$. Let $x_0 < x_1$ and $x_{n+1} > x_n$ be such that the support of $u$ is contained in $[x_0, x_{n+1})$. For $k = 0, \ldots, n$, set $I_k = (x_k, x_{k+1})$ and denote the length of $I_k$ by $l_k$. Denote by $\omega_k$ the oscillation of $u$ on $I_k$: $\omega_k(\eta) = \sup\{\|u(x) - u(y)\|: x, y \in I_k, |x - y| \leq \eta\}$. It is a nondecreasing function with $\omega_k(0^+) \leq \delta/2$. We choose $\varepsilon < \min_k l_k$. We can divide $I_k$ into subintervals with lengths larger than $\varepsilon$ but smaller than $2\varepsilon$. On each subinterval of $I_k$ we replace $u(\cdot)$ by its mean value. We claim that the function $u^{\varepsilon, \delta}$ thus defined satisfies the result of the lemma. Indeed, the integral $F_u(x) - F_{u^{\varepsilon, \delta}}(x)$ has two contributions. The first one comes from integrating over $u^{\varepsilon, \delta}$-steps that do not contain $x$, and vanishes because by construction $u$ and $u^{\varepsilon, \delta}$ produce the same integral over each whole step. The second contribution is of the form $\int_\delta^\beta [u(y) - u^{\varepsilon, \delta}(y)] \mathrm{d}y$ where $x \in [\alpha, \beta]$ and $[\alpha, \beta] \subset I_k$ is one of the steps. The latter contribution is bounded by $2\varepsilon \omega_k(2\varepsilon)$. To obtain the result we choose $\varepsilon$ small enough to have $\sup_k \omega_k(2\varepsilon) < \delta/2$.

**Proof of Theorem 3.1.** Assumption 2 for both semigroups $T$ and $U$ reduces the problem to $u_0 \in B_+$ with compact support. Moreover, a measurable profile with bounded support can be $A$-approximated by a profile with bounded support and finite variation. Therefore, using assumption 3 we may further restrict to $u_0(\cdot)$ with compact support and finite variation, which we now assume. Set $u_t(\cdot) = T_t u_0(\cdot)$ and $u(\cdot, t) = U_t u_0(\cdot)$. Since the constant 0 is a Riemann entropy solution, assumptions 1 and 2 imply that $u_t$ and $u(\cdot, t)$ have compact support in space expanding in time with maximum speed $v'$ defined in Lemma 3.4. We set $A(t) = A(u_t(\cdot), u(\cdot, t))$. We fix some $\delta > 0$. The total variation of $u(\cdot, t)$ is bounded by that of $u_0(\cdot)$ (see Chapter 2 of Serre, 1996). Hence we may approximate $u(\cdot, t)$ with $u^{\varepsilon}(\cdot, t)$ given by Lemma 3.6, where for notational simplicity we omit dependence on $\delta$. This requires $\varepsilon < \varepsilon(t)$ where $\varepsilon(t)$ depends on $u(\cdot, t)$.
As in Glimm’s scheme, we locally in time replace the current profiles at time \( t \) by the piecewise constant one \( u^\varepsilon(.,t) \). Starting from the latter \( T \) and \( U \) coincide for some time, so the increase of \( \Delta \) will come essentially from the replacement error. Precisely, set \( u^\varepsilon(.,t) := U_t u^\varepsilon(.,t) \) and \( u^\varepsilon(.,s) := T_s u^\varepsilon(.,t) \). By Lemma 3.4, we know that \( u^\varepsilon(.,s) = u^\varepsilon(.,t) \) for \( 0 \leq s \leq \varepsilon/2v' \). Then, for these values of \( s \):

\[
\Delta(t + s) = \Delta(u_{t+s}(.), u(., t + s)) \\
\leq \Delta(u_{t+s}(.), u^\varepsilon(.,t)) + \Delta(u^\varepsilon(.,t), u(., t + s)) \\
\leq \Delta(u_{t}(.), u^\varepsilon(.,t)) + \Delta(u^\varepsilon(.,t), u(., t)) \\
\leq \Delta(u_{t}(.), u(., t)) + \Delta(u(., t), u^\varepsilon(.,t)) + \Delta(u^\varepsilon(.,t), u(., t)) \\
\leq \Delta(t) + 2\delta \varepsilon,
\]

where we have used semigroup property and assumption 4. Thus we have proved that for all \( t > 0 \) there exists \( \varepsilon(t) > 0 \) such that for any \( \varepsilon < \varepsilon(t) \):

\[
\Delta(t + \varepsilon/2v') \leq \Delta(t) + 2\delta \varepsilon. \tag{13}
\]

Since \( \Delta(0) = 0 \) and \( \Delta(t) \) is continuous by assumption 3, (13) implies \( \Delta(t) \leq 4v' \delta t \) for every \( t \geq 0 \). As this holds for any fixed \( \delta > 0 \), we conclude that \( \Delta(.) \) is identically 0.

For a finite particle configuration \( \eta \) and an integrable density profile \( u(.) \) we extend the definition of \( \Delta^N \) by

\[
\Delta^N(\eta, u(.)) = \sup_{x \in \mathbb{Z}} |N^{-1} F_{\eta}(x) - F_{u(.)}(x/N)|.
\]

It is not difficult to prove that, if the sequence \( \eta^N \) has bounded support and density profile \( u(.) \in \mathcal{B}_+ \), then it has bounded mass and \( \Delta^N(\eta^N, u(.)) = o_N(1) \) in probability. Note also that by Lemma 3.1, the sequence \( \eta^N_{Nt} \) still has bounded support for every \( t > 0 \).

**Proof of Theorem 3.2.** The proof mimics that of Theorem 3.1. As previously we set \( u(.,t) := U_t u_0(.) \) for the entropy solution. We consider the particle system starting from a sequence of configurations \( \eta^N \) with density profile \( u_0(.) \). The particle configuration at time \( Nt \) will play the role of \( u_t \). Proceeding as in the proof of Theorem 3.1, we may restrict to the case where \( u_0(.) \) has bounded support and finite variation, and \( \eta^N \) has bounded support. This is done using assumption 2 of Theorem 3.1 and Lemma 3.3 on the one hand, assumption 3 and macroscopic stability on the other hand. Define \( \Delta^N(t) = \Delta^N(\eta^N_{Nt}, u(t,.)) \). We will show that the sequence of processes \( \Delta^N(.) \) converges in probability to the identically null process w.r.t. local uniform topology on \( C^0([0; \infty); \mathbb{R}) \), which implies the hydrodynamic limit. Lemma 3.2 and assumption 3 of Theorem 3.1 imply that the sequence is tight w.r.t. this topology, so every subsequential weak limit is a process \( \Delta(.) \) with a.s. continuous paths. Moreover it a.s. satisfies \( \Delta(0) = 0 \), which simply expresses convergence to initial profile.

We fix \( \delta > 0 \) and define \( u^\varepsilon(.,t) \) and \( u^\varepsilon(.,s) \) as in the proof of Theorem 3.1 for \( \varepsilon < \varepsilon(t) \). We introduce a new sequence of initial configurations \( \eta^{N,\varepsilon,t} \) with bounded
support and density profile \( u^\varepsilon(.,t) \). From these configurations we start new systems denoted by \( \eta_{N_0}^{N_0,\varepsilon,t} \) for \( s \geq 0 \). Next, we write

\[
\Delta^N(t+s) = \Delta^N(\eta_{N(t+s)}, u(.,t+s))
\]

\[
\leq \Delta^N(\eta_{N(t+s)}, \eta_{N_0}^{N_0,\varepsilon,t}) + \Delta^N(\eta_{N_0}^{N_0,\varepsilon,t}, u^\varepsilon(.,s)) + \Delta(u^\varepsilon(.,s), u(.,t+s))
\]

\[
\leq o_N(1) + \Delta^N(\eta_{N_0}^{N_0,\varepsilon,t}, u(.,t)) + \Delta(u(.,t), u^\varepsilon(.,t)) + \Delta^N(u^\varepsilon(.,t), \eta_{N_0}^{N_0,\varepsilon,t})
\]

\[
+ \Delta^N(\eta_{N_0}^{N_0,\varepsilon,t}, u^\varepsilon(.,s)) + \Delta(u^\varepsilon(.,t), u(.,t))
\]

\[
= o_N(1) + \Delta^N(t) + \delta \varepsilon + o_N(1) + o_N(1) + \delta \varepsilon.
\]

In the second inequality we used macroscopic stability for the first term and assumption 4 of Theorem 3.1 for the last one. In the last equality \( \delta \varepsilon \) comes from Lemma 3.6, and the two additional \( o_N(1) \) from the fact that \( \eta_{N_0}^{N_0,\varepsilon,t} \) has bounded support and density profile \( u^\varepsilon(.,s) \). Passing to the limit we obtain that, for every \( \delta > 0 \), a.e. path of \( \Delta(\cdot) \) satisfies (13) with \( V' \) instead of \( v' \), and thus \( \Delta(t) \leq 4V't \) for all \( t > 0 \). We conclude taking a vanishing sequence of \( \delta \)'s such that a.e. path of \( \Delta(\cdot) \) satisfies this simultaneously for all these \( \delta \)'s.

4. Examples

In this section we will review \( k \)-step exclusion processes, \( k \)-range queuing processes (that we introduce here) and misanthrope processes. We will spend a little more time on \( k \)-exclusion since it has been the example that motivated us. In all subsequent examples, the generator is of the form

\[
Lf(\eta) = \sum_{x,y \in \mathbb{Z}} c(x,y,\eta)[f(\eta^{x,y}) - f(\eta)]
\]  

(14)

for local functions \( f \), where \( \eta^{x,y} \) denotes the new state after a particle has jumped from \( x \) to \( y \), and \( c(x,y,\eta) \) is the rate at which a particle jumps from \( x \) to \( y \) in the configuration \( \eta \). Translation invariance and finite range imply that \( c(x,y,\eta) = c(0,y-x,\tau_x \eta), \) and that \( c(0,z,.) \) are local functions of \( \eta \) that vanish for all but a finite set of \( z \). Using (14), we can rewrite (4) as

\[
j(\eta) = \sum_{x \leq 0, y > 0} c(x,y,\eta) - \sum_{x > 0, y \leq 0} c(x,y,\eta).
\]

Using translation invariance of \( v_z \), we see that

\[
G(u) = v_u \left[ \sum_{z \in \mathbb{Z}} z c(0,z,\eta) \right].
\]  

(15)
4.1. The $k$-step exclusion process

In his paper Liggett (1980) introduced a Feller nonconservative approximation of the long range exclusion process to study the latter. The $k$-step exclusion process is a conservative version of this dynamics, defined and studied in Guiol (1999). It is described in the following way.

Let $k \in \mathbb{N}^* := \{1, 2, \ldots \}$, $X := \{0, 1\}^\mathbb{Z}$ be the state space, and let $\{X_n\}_{n \in \mathbb{N}}$ be a Markov chain on $\mathbb{Z}$ with transition matrix $p(\ldots)$ and $P^0(X_0 = x) = 1$. Under the mild hypothesis $\sup_{y \in \mathbb{Z}} \sum_{x \in \mathbb{Z}} p(x, y) < +\infty$, the $k$-step exclusion process with jump kernel $p(\ldots)$ can be defined by (14), with

$$c(x, y, \eta) = \eta(x)(1 - \eta(y))\mathbf{E}^{x} \left[ \prod_{i=1}^{\sigma_y - 1} \eta(X_i), \sigma_x \leq \sigma_y \leq \sigma_x + k \right],$$

where $\sigma_y = \inf\{n \geq 1 : X_n = y\}$ is the first (nonzero) arrival time to site $y$ of the chain starting at site $x$. In words if a particle at site $x$ wants to jump it may go to the first empty site encountered before returning to site $x$ following the chain $X_n$ (starting at $x$) provided it takes less than $k$ attempts; otherwise the movement is cancelled.

When $k = 1$, $(\eta_t)_{t \geq 0}$ is the classical simple exclusion process. An important property of $k$-step exclusion is that it is an attractive process.

Let $\mathcal{I}_k$ be the set of invariant measures for $(\eta_t)_{t \geq 0}$ on $X$. When $p(x, y)$ is an irreducible random walk on $\mathbb{Z}$, Guiol (1999) proved that

$$(\mathcal{I}_k \cap \mathcal{L})_c = \{v_x : x \in [0, 1]\},$$

where $v_x$ is the Bernoulli product measure with constant density $x$. Then the results of Sections 2 and 3 apply. In the totally asymmetric case, we have $c(x, y, \eta) = 1$ if $\eta(x) = 1$, $y - x \in \{1, \ldots, k\}$ and $y$ is the first nonoccupied site to the right of $x$; otherwise $c(x, y, \eta) = 0$. Then (15) yields the flux $G_k(u) = \sum_{j=1}^{k} j u^j (1 - u)$. More generally, in the nearest-neighbor case with jump probabilities $p$ and $q$ to the right and left, the flux function reads:

$$G_k(u) = (1 - u) \left\{ (p - q)u + 2(p^2 - q^2)u^2 \right. \right.$$

$$+ \sum_{j=3}^{k} j(p^j - q^j) \left( 1 + \sum_{l=1}^{[(k-j)/2]} \binom{j-2+l-1}{l} (pq)^l \right) u^j \right\},$$

where $[.]$ stands for the integer part.

The “drift terms” are factors of $(p^j - q^j)u^j$ only. Moreover, on a path of $j+1$ sites (e.g. from 0 to $j$, site $j$ empty) one can perform $l$ loops (e.g. a path like $x+1 \rightarrow x \rightarrow x+1$) between the $j-2$ intermediate pairs of sites (once the particle returns to 0 or reaches $j$ it stays there) provided $l$ does not exceed $(k-j)/2$. 

Fig. 1. Hydrodynamic behavior of the 2-step exclusion with Riemann initial conditions, graph of \( u(x,t;1) \).

(a) Rarefaction fan \( \lambda = 0.025, \rho = 0.13 < \frac{1}{6}(\rho^* = 0.185) \); (b) decreasing shock \( \lambda = 0.15, \rho = 0.13 \); (c) contact discontinuity \( \lambda = 0.4, \rho = 0.13 \); (d) rarefaction fan \( \lambda = 0.7, \rho = 0.2 > \frac{1}{6}(\rho^* = 0.15) \); (e) increasing shock \( \lambda = 0.17, \rho = 0.2 \); (f) contact discontinuity \( \lambda = 0.01, \rho = 0.2 \).

**Remark.** Let \( k \) go to infinity and denote by \( G_\infty \) the limiting flux function in the totally asymmetric case:

\[
G_\infty(u) = \frac{u}{1-u}.
\]

That case corresponds to the totally asymmetric long range exclusion process. The resulting equation is simpler because the flux function \( G_\infty \) is strictly convex, but some care is needed because this process is not Feller. The hydrodynamics here should follow from the arguments of Aldous and Diaconis (1995) for the Hammersley’s process.

Below we present the solution of the Riemann problem for the totally asymmetric 2-step exclusion process. Then we show that the flux function of a totally asymmetric \( k \)-step exclusion process has only one inflexion point. We finally show how one can generate a flux with many inflexion points by superimposing several \( k \)-step exclusion processes.

### 4.1.1. The example of totally asymmetric 2-step exclusion

Its flux function \( G_2(u) = u + u^2 - 2u^3 \) is strictly convex in \( 0 \leq u < \frac{1}{6} \) and strictly concave in \( \frac{1}{6} < u \leq 1 \). For \( w < \frac{1}{6}, w^* = (1-2w)/4 \), and for \( w > \frac{1}{6}, w_\ast = (1-2w)/4 \); \( h_1(x) = (\frac{1}{6}) (1-\sqrt{7-6x}) \) for \( x \in (-\infty, \frac{7}{6}) \), and \( h_2(x) = (\frac{1}{6}) (1+\sqrt{7-6x}) \) for \( x \in (\frac{7}{6}, +\infty) \).

Fig. 1 shows the six possible behaviors of the (self-similar) solution \( u(v,1) \). Cases (a) and (b) present, respectively, a rarefaction fan with increasing initial condition and a preserved decreasing shock. These situations as well as cases (c) and (f) cannot occur for simple exclusion. Observe also that \( \rho \geq \frac{1}{6} \) implies \( \rho_\ast \leq 0 \), which leads only to cases (d), (e), and excludes case (f) (going back to a simple exclusion behavior).
4.1.2. Uniqueness of inflexion point for the totally asymmetric $k$-step exclusion

**Lemma 4.1.** The function $G_k$ has at most one inflexion point on $(0, +\infty)$.

**Proof.** The second derivative of flux can be written as
\[
G_k''(u) = \sum_{j=2}^{k} j(j-1)u^{j-2} - k^2(k+1)u^{k-1}.
\]

We claim that a function of this form has no more than one zero on $(0, \infty)$. Here is the argument: Let $u_0$ be the smallest zero of the polynomial above. Then we have
\[
\sum_{j=2}^{k} j(j-1)u_0^{j-2} = k^2(k+1)u_0^{k-1}.
\]

If $u > u_0$
\[
\sum_{j=2}^{k} j(j-1)u^{j-2} < \left( \frac{u}{u_0} \right)^k \sum_{j=2}^{k} j(j-1)u_0^{j-2}
\]
\[
= \left( \frac{u}{u_0} \right)^{n-2} (k^2(k+1)u_0^{k-1}) < \left( \frac{u}{u_0} \right)^{n-1} (k^2(k+1)u_0^{k-1})
\]
\[
= k^2(k+1)u^{k-1}.
\]

\[\square\]

4.1.3. Superposition of $k$-step exclusion processes

A simple way to construct systems with several (but finitely many) inflexion points is to superimpose different $k$-step exclusion processes (with possibly different values of $k$). Since the Bernoulli product measures are invariant for each generator, they are still invariant for the process governed by the sum of these generators. One can show that the resulting system is attractive and, following Guiol (1999), that these measures still satisfy (2) (the latter is indeed connected with an irreducibility property which is maintained a fortiori if one increases the set of possible transitions). Therefore the results of Sections 2 and 3 apply to such systems, and the resulting flux is simply the sum of individual fluxes (Fig. 2).

In the picture below we have superimposed two nearest-neighbor asymmetric $k$-step exclusion processes: a 2-step with $p=0.9$ ($q=0.1$) and a 3-step with $p=0.2$ ($q=0.8$). The flux functions just sum giving rise to two inflexion points $x_1 \simeq 0.0793$ and $x_2 \simeq 0.6162$.

4.2. A $k$-range queuing process

This example is related to the “Tagged Pushing Particle” that will be introduced in the next section.

The $k$-range queuing process can be described informally as follows: It is an infinite server system (FIFO queues) with state space $\mathbb{N}^2$. There is a server on each site of $\mathbb{Z}$,
whose service times are independent, with an exponential distribution of rate 1. When a server \( x \) has finished with a client, the latter then chooses to move to a server \( y \) according to the following rules:

(i) \( |y - x| \leq k \);
(ii) all servers between \( x \) and \( y \) are non-occupied;
(iii) site \( y \) is chosen with the probability that a nearest-neighbor random walk, with absorbing barrier at \( x \), starting from \( x \) ends up at \( y \) in less than \( k \) steps.

Its generator is given by (14) with

\[
c(x, y, \eta) = 1_{\{\eta(x) > 0\}} E^x \left[ \prod_{n=1}^{\sigma_y - 1} 1_{\{\eta(X_n) = 0\}}, \sigma_y \leq k \right],
\]

where \((X_n)_{n \geq 0}\) is a nearest-neighbor simple random walk, with \(P^x(X_0 = x) = 1\), absorbing barrier at \( x \) and \( \sigma_y = \inf\{n > 0: X_n = y\}\) with convention \(\inf\emptyset = +\infty\).

Loosely speaking a client can drop the free servers (up to \( k \)).

When \( k = 1 \) it corresponds to the zero range process (see next example). The \( k \) range queuing process is attractive and has a one-parameter family of ergodic invariant product measures (cf. Theorem 5.1 below).

The flux for the totally asymmetric \( k \)-range queuing process reads:

\[
G(u) = \frac{u}{1 + u} \sum_{j=1}^{k} j \left( \frac{1}{1 + u} \right)^{j-1}.
\]

It provides another example with at most one inflection point.

4.3. The misanthrope process

This process was introduced in Cocozza-Thivent (1985). It has state space \( X = \mathbb{N}^\mathbb{Z} \), or \( \{0, \ldots, N\}^\mathbb{Z} \) if \( N \) defined in (v) below is finite. Let \( p(x, y) \) be an irreducible random walk on \( \mathbb{Z} \), \( \gamma = \sum_{x \in \mathbb{Z}} x p(0, x) \) denote its drift. Its generator is of the form (14) with \( c(x, y, \eta) = b(\eta(x), \eta(y)) \), where:

(i) \( b(\ldots) \) is a bounded function defined on \( \mathbb{N} \times \mathbb{N} \), positive on \( \mathbb{N} - \{0\} \times \mathbb{N} \), equal to 0 on \( \{0\} \times \mathbb{N} \).
(ii) attractivity: for every \( m \in \mathbb{N} \), \( n \mapsto b(n, m) \) is nondecreasing, and for every \( n \in \mathbb{N} \), \( m \mapsto b(n, m) \) is nonincreasing.

(iii) for any \( i \geq 1, j \geq 0 \), \( b(., .) \) satisfies:
\[
b(n, m)b(m + 1, 0)b(1, n - 1) = b(m + 1, n - 1)b(n, 0)b(1, m);
\]
(16)

(iv) and the gradient property:
\[
\text{for } n \geq 0, m \geq 0, \quad b(n, m) - b(m, n) = b(n, 0) - b(m, 0).
\]
(17)

(v) For any \( n \geq \mathcal{N} \), \( b(n, 0) = b(\mathcal{N}, 0) \); where \( \mathcal{N} = \inf \{ m \in \mathbb{N} : b(1, m) = 0 \} \) is the maximum occupation number, with convention \( \inf \emptyset = +\infty \). We define
\[
a_0 = a_1 = 1, \quad a_n = \prod_{i=1}^{n-1} \frac{b(1, i)}{b(i + 1, 0)}.
\]

Let \( Z(\varphi) = \sum_{n \geq 0} a_n \varphi^n \) be a partition function and let \( q_\infty = \sup_{n \in \mathbb{N}} b(n, 0)/b(1, n - 1) \), the function \( A : [0, q_\infty) \to [0, \mathcal{N}) \) defined by
\[
A(\varphi) = \frac{\sum_{n \geq 0} a_n \varphi^n}{\sum_{n \geq 0} a_n \varphi^n} = \varphi \frac{Z'(\varphi)}{Z(\varphi)}
\]
is increasing thus has a well-defined inverse \( A^{-1} \); the product measure \( v_z \) with marginals
\[
v_z \{ \eta \in X : \eta(x) = n \} = \frac{a_n \varphi^n}{Z(\varphi)}.
\]

where \( \varphi = A^{-1}(z) \) satisfies \( v_z(\eta(x)) = z \). It is proven in Cocozza-Thivent (1985) that under hypotheses (16) and (17) the measures \( \{ v_z \}_z \) satisfy (2). Following (15) the flux function is given by
\[
G(u) = \gamma \int \left( \frac{d v_u(\eta)}{b(\eta(x), \eta(y))} \right).
\]
(18)

We now review some examples of misanthropes processes. For simplicity of notation we shall assume \( \gamma = 1 \).

### 4.3.1. The zero range process

This special case of misanthropes processes is one of the most studied conservative, attractive, Feller processes, we refer to Andjel (1982) for more details.

It has state space \( X = \mathbb{N}^2 \). It corresponds to \( b(n, m) = g(n) \), for a function \( g \) that does not need to be bounded, but should satisfy \( \sup_{k \in \mathbb{N}} (g(k + 1) - g(k)) < \infty \), \( \mathcal{N} = +\infty \). \( G \) is always an increasing function. For the zero range process with constant rate e.g. \( g(k) = 1_{\{k \geq 0\}} \) it reads
\[
G(u) = \frac{u}{1 + u}.
\]

It is not known whether \( G \) is always concave for more general choices of \( g \), which is why this assumption had to be added in Andjel and Vares (1987).
4.3.2. Two classes of examples

The following examples are derived from Bahadoran (1997, p. 10):

(i) Product jump rate

Let \( b(n,m) = g(n)(a - bg(m)) \) where \( a > 0 \), \( b \) is a nonnegative repulsion factor and \( g \) is a nondecreasing Lipschitz function such that \( g(0) = 0 \), \( \lim_{m \to +\infty} bg(m) \leq a \).

(a) We consider the particular case \( a = b = 1 \) and \( g(n) = 1 - (n + 1)^{-1} \), then \( N = +\infty \) and \( q_\infty = +\infty \). This way, for \( n \geq 2 \),

\[
a_n = \left( \frac{1}{2} \right)^n \frac{n + 1}{n!}, \quad Z(\varphi) = \left( \frac{\varphi}{2} + 1 \right) \exp\left( \frac{\varphi}{2} \right),
\]

\[
A(\varphi) = \frac{\varphi(4 + \varphi)}{2(2 + \varphi)}, \quad A^{-1}(x) = -2 + x + \sqrt{4 + x^2}.
\]

The flux function

\[
G(u) = \frac{2(-2 + u + \sqrt{4 + u^2})}{(u + \sqrt{4 + u^2})^2}
\]

has one inflexion point, at 2.781618048.

(b) 2-exclusion misanthrope process

Hydrodynamic behavior of a totally asymmetric, nearest neighbor \( K \)-exclusion process with constant jump rates was studied in Seppäläinen (1999). Here we consider a 2-exclusion process with nonconstant rates but product invariant measures.

This process has state space \( X = \{0, 1, 2\}^Z \): in the previous example take \( a = 2 \), \( b = 1 \) and \( g(n) = n \land 2 \). Then \( N = 2 \) and \( q_\infty = +\infty \). The flux function follows immediately:

\[
G(u) = v_\eta[\eta(x)(2 - \eta(y))] = u(2 - u)
\]

is a strictly concave function.

(ii) Rates with a finite number of distinct values

In this case the jump rates have the form

\[
b(n,m) = b(n \land 2, m \land 2)
\]

with

\[
b(1,0) = \alpha_1, \quad b(2,0) = \alpha_2, \quad b(1,1) = \beta_1, \quad b(1,2) = \beta_2.
\]

To satisfy (16) and (17) we need

\[
b(2,1) = \beta_2 + \alpha_2 - \alpha_1, \quad \beta_1 b(2,2) = \beta_2 b(2,1).
\]

For the process to be attractive we need

\[
\alpha_1 \leq \alpha_2, \quad \beta_1 \geq \beta_2, \quad \beta_2 + \alpha_2 \geq \beta_1 + \alpha_1
\]

hence we have \( N = +\infty \) and \( q_\infty = x_2/\beta_2 \); the partition function reads

\[
Z(\varphi) = \frac{(\beta_1 - \beta_2)\varphi^2 + (\alpha_2 - \beta_2)\varphi + \alpha_2}{\alpha_2 - \beta_2 \varphi}
\]
and

$$A(\varphi) = \varphi \frac{-\beta_2(\beta_1 - \beta_2)\varphi^2 + \varphi_2(\beta_1 - \beta_2)\varphi + \varphi_2^2}{[(\beta_1 - \beta_2)\varphi^2 + (\varphi_2 - \beta_2)\varphi + \varphi_2][\varphi_2 - \beta_2\varphi]}.$$ 

Observe that a small change in one coefficient could produce a strictly different behavior for hydrodynamics:

(a) taking the values $\varphi_1 = 4$, $\varphi_2 = 25$, $\beta_1 = \beta_2 = 5$, then

$$Z(\varphi) = \frac{5 + 4\varphi}{5 - \varphi}$$

and

$$A(\varphi) = \frac{25\varphi}{25 + 15\varphi - 4\varphi^2}; \quad A^{-1}(x) = \frac{5}{8} \left[ 3 - \frac{5}{x} + \sqrt{25 - \frac{30}{x} + \frac{25}{x^2}} \right].$$

The corresponding flux function

$$G(u) = \frac{2A^{-1}(u)(50 + 105A^{-1}(u) + 22(A^{-1}(u))^2)}{(5 + 4A^{-1}(u))^2}$$

has one inflexion point on $(0, +\infty)$.

(b) keeping the same values of the parameters but $\beta_1 = 9$ then

$$Z(\varphi) = \frac{(2\varphi + 5)^2}{5(5 - \varphi)}$$

and

$$A(\varphi) = \frac{\varphi(25 - 2\varphi)}{(2\varphi + 5)(5 - \varphi)}; \quad A^{-1}(x) = \frac{5(x + \sqrt{25 - 18x + 9x^2 - 5})}{4(x - 1)}.$$ 

We obtain that the flux function

$$G(u) = \frac{5}{2}(45u^2 + 37u\sqrt{25 - 18u + 9u^2} - 238u + 193$$

$$- 37\sqrt{25 - 18u + 9u^2}(-5 + u + \sqrt{25 - 18u + 9u^2})$$

$$(-7 + 3u + \sqrt{25 - 18u + 9u^2})^{-1}(37 - 30u)$$

$$- 7\sqrt{25 - 18u + 9u^2} + 9u^2 + 3u\sqrt{25 - 18u + 9u^2})^{-1}$$

has two inflexion points in $(0, +\infty)$, namely, $0.5579892097$ and $19.07844811$.

5. Asymptotic behavior of a tagged particle

We introduce here an interpretation of the $k$-step exclusion dynamics valid for the one dimensional nearest neighbor process. For simplicity we focus on the totally asymmetric case but our results (up to more tedious formulas) are valid for the nearest
neighbor asymmetric case. Up to now we considered that a particle might jump from $x$ to the first empty site in $\{x + 1, \ldots, x + k\}$. But, taking advantage that particles are indistinguishable, if we want to keep the initial particles’ order, we could equally say that the particle at $x$ pushes the “pack” of ($\leq k$) neighboring particles in front of it, each one moving of one unit to the right.

Similarly, we define a Tagged “pushing” particle, and the generator of the $k$-step exclusion process as seen from this tagged pushing particle reads

$$\tilde{L}_k f(\eta)$$

$$= \sum_{i=0}^{k-1} \sum_{x \neq 0, -1, \ldots, -(i+1)} \left[ \prod_{j=0}^{i} \eta(x+j) \right] (1 - \eta(x+i+1)) [f(\eta^{x+i+1}) - f(\eta)] + \sum_{n=1}^{k} \left[ \prod_{m=1}^{n-1} \eta(m) \right] (1 - \eta(n)) [f(\tau_1 \eta^{0,n}) - f(\eta)]$$

$$+ \sum_{n=1}^{k-1} \sum_{l=1}^{k-n} \left[ \prod_{m=-l}^{-1} \eta(m) \prod_{i=1}^{m-1} \eta(i) \right] (1 - \eta(n)) [f(\tau_1 \eta^{-l,n}) - f(\eta)].$$

To be clearer, let us write and comment it for $k = 2$.

$$\tilde{L}_2 f(\eta) = \sum_{x \neq 0, -1} \eta(x)(1 - \eta(x+1)) [f(\eta^{x+1}) - f(\eta)]$$

$$+ (1 - \eta(1)) [f(\tau_1 \eta^{0,1}) - f(\eta)]$$

$$+ \sum_{x \neq 0, -1, -2} \eta(x)\eta(x+1)(1 - \eta(x+2)) [f(\eta^{x+2}) - f(\eta)]$$

$$+ \eta(1)(1 - \eta(2)) [f(\tau_1 \eta^{0,2}) - f(\eta)]$$

$$+ \eta(-1)(1 - \eta(-1)) [f(\tau_1 \eta^{-1,1}) - f(\eta)].$$

Parts (19) and (21) involve sites away from the origin, (19) for simple exclusion, (21) for a “strictly” 2 steps exclusion. Part (20) corresponds to the “classical” tagged particle for simple exclusion, (22) describes “pushing” by the tagged particle; in (23) the tagged particle is “pushed” by another one.

In the following we extend to the $k$-step exclusion process as seen from the pushing particle some result of Ferrari (1986). We refer to this article for notations.

**Theorem 5.1.** The Palm measure $\hat{\nu}_{\alpha}$ of $\nu_{\alpha}$ (i.e. the measure on $X$ defined by $\hat{\nu}_{\alpha}(\cdot) = \nu_{\alpha}(\cdot|\eta(0) = 1)$) is extremal invariant for the $k$-step exclusion process as seen from a tagged pushing particle.
Sketch of Proof. Denote by $\hat{S}_k(t)$ the semigroup associated to the $k$-step exclusion process as seen from a tagged pushing particle. The invariance of $\hat{v}_x$ comes from the invariance of $\hat{v}_x$ for $k$-step exclusion and Ferrari (1986), Theorem 2.3 which shows that for any $\mu \in \mathcal{S}$ and all $t > 0$, $\mu \hat{S}_k(t) = \hat{\mu} \hat{S}_k(t)$.

To obtain extremality we follow the proof of Ferrari (1986), Theorem 3.4: The one dimensional asymmetric simple exclusion process as seen from the tagged particle is isomorphic to the one dimensional asymmetric zero range process with constant rate 1 (see also the references in Ferrari, 1986), which has a one-parameter family of extremal invariant measures (Andjel, 1982, Theorem 1.11). In a similar way, the asymmetric $k$-step exclusion process as seen from the tagged pushing particle is isomorphic to a nearest neighbor $k$-range queuing process (see previous section). The proof that the latter has also a one-parameter family of extremal invariant measures is a straightforward adaptation of Andjel (1982, Theorem 1.11).

Theorem 5.2. Law of large numbers for the tagged pushing particle.

For a $k$-step exclusion process with initial distribution $\hat{v}_x$, if $Y(t)$ denotes the position at time $t$ of a tagged pushing particle starting at the origin then

$$\lim_{t \to \infty} \frac{Y(t)}{t} = (1 - \alpha) \left( \sum_{j=1}^{k} j x^{j-1} \right) P_{\hat{v}_x} \ a.s.$$

Proof. Using the notation of Ferrari (1992, pp. 41–43) we define the instantaneous increment of the position of the tagged pushing particle by

$$\psi(\eta) := \lim_{h \to 0} \frac{E(Y(t + h) - Y(t)|Y_t = x)}{h}$$

$$= (1 - \eta(x + 1)) + \eta(x - 1)(1 - \eta(x + 1)) + \eta(x + 1)(1 - \eta(x + 2)) + \cdots$$

$$+ \eta(x - k + 1) \cdots \eta(x - 1)(1 - \eta(x + 1)) + \cdots$$

$$+ \eta(x + 1) \cdots \eta(x + k - 1)(1 - \eta(x + k)).$$

So

$$\int \psi \, dv_x = (1 - \alpha) \left( \sum_{j=1}^{k} j x^{j-1} \right).$$

Furthermore $\lim_{h \to 0} E_{\hat{v}_x}(Y(t+h) - Y(t))^2/h < +\infty$ so that by Theorem 5.1 the conditions of Ferrari (1992, Theorem 9.2) are satisfied, which gives the result.

Remark. Referring to results of Guiol (1999), we notice that a tagged pushing particle $Y(t)$ behaves as a regular tagged particle in $k$-step exclusion. Indeed, intuitively, the “regular” tagged particle can make long jumps, so is expected to move faster, but it cannot be pushed; and the rate at which the tagged pushing particle moves compensates exactly those long jumps.
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Appendix

Proof of Lemma 2.2. \( h_c(v - 0) \) and \( h_c(v + 0) \) are, respectively, the smallest and greatest \( x \) such that \( H_c(x) = v \). Therefore, since \( H_c \) is nondecreasing, \( h_c(v - 0) \) and \( h_c(v + 0) \) are the smallest and greatest \( x \) where global minimum of \( G_c \) is attained. And because \( G_c \) is convex, it must be constant between these two values. This proves the last statement.

Now observe that

The function \( G_c(x) - vx \) is the lower convex envelope of \( G(x) - vx \). \( G_c(x) - vx \) is convex; satisfies \( G_c(x) - vx \leq H(x) \leq G(x) - vx \), but differs from \( G_c(x) - vx \). This contradicts (24). Therefore

\[
\inf_{x \in [\lambda, \rho]} (G(x) - vx) = \inf_{x \in [\lambda, \rho]} (G_c(x) - vx),
\]

which proves the first statement.

Denote by \( x_-(v) \) and \( x_+(v) \) the smallest and greatest global minimum of \( G(x) - vx \). We claim that every global minimum \( x' \) for \( G(x) - vx \) is also one for \( G_c(x) - vx \). Otherwise, we would have \( G(x') - vx' \geq G_c(x') - vx' \geq \inf_{x \in [\lambda, \rho]} G_c(x) - vx \), which contradicts (25). Thus \( x_-(v) \geq h_c(v - 0) \) and \( x_+(v) \leq h_c(v + 0) \). It remains to prove that these inequalities cannot be strict. Assume for instance that \( x_+(v) < h_c(v + 0) \). Denote by \( m \) the minimum value of both \( G(x) - vx \) and \( G_c(x) - vx \). \( G_c(x) - vx \) has constant value \( m \) on \( [x_+(v), h_c(v + 0)] \). Let \( \beta = (x_+(v) + h_c(v + 0))/2 \). By definition of \( x_+(v) \), we have

\[
\inf_{x \in [\beta, \rho]} G(x) - vx := m' > m.
\]
Now the function defined by

\[
N(x) = \begin{cases} 
G_c(x) - vx & \text{if } \lambda \leq x < \alpha_+(v), \\
m & \text{if } \alpha_+(v) \leq x < \beta, \\
\max \{G_c(x) - vx; m + (m' - m) \left[ \frac{x - \beta}{\rho - \beta} \right] \} & \text{if } \beta \leq x \leq \rho,
\end{cases}
\]

is convex, satisfies \( G_c(x) - vx \leq N(x) \leq G(x) - vx \), but differs from \( G_c(x) - vx \), which contradicts (24). A similar argument shows that \( \alpha_-(v) = h_c(v - 0) \). \( \square \)

**Proof of Corollary 2.1.** Assume first that \( v \in \Sigma_{\text{low}}(G) \). By Lemma 2.2, the points \((h_c(v \pm 0), G_c(h_c(v \pm 0)))\) lie on the graphs of \( G \) and \( G_c \). By convexity, \( G_c(x) - vx \) is constant between the two global minima \( h_c(v \pm 0) \), so the graph of \( G_c \) between these points is linear with slope \( v \). It is thus also a chord for \( G \).

If \( v \not\in \tilde{\Sigma}_{\text{low}}(G) \), by definition of \( h_c \), there is a neighborhood \( I \) of \( h_c(v) \) and a neighborhood \( J = H_c(I) \subset (\Sigma_{\text{low}}(G))^c \) of \( v \) such that \( H_c \) is strictly increasing from \( I \) to \( J \) with continuous inverse \( h_c \) on \( J \). Thus \( G_c \) is strictly convex on \( I \). For \( w \in J \), \( h_c(w) \) is the unique global minimum of \( G_c(x) - wx \) and \( G(x) - wx \) by Lemma 2.2. Therefore the graphs of \( G \) and \( G_c \) coincide on \( I \). \( \square \)

**Proof of Lemma 3.1.** In this and the next proof we assume that the generator is of the form (14) with bounded \( c \). Let \( r \in \mathbb{N}^* \) be such that \( c(0, z, \eta) \) vanishes for \( |z| > r \) and depends only on \( \eta \) through sites \( -r \leq u \leq r \). We can couple \( \eta \) and \( \zeta \) in such a way that, whenever a jump occurs for one of the systems from \( u \) to \( v \), and \( c(u, v, \eta) = c(u, v, \zeta) \), the jump must also occur for the other system. Note that this coupling does not require attractivity. Now define the trajectory \( X^x_t \) starting at \( x \) as follows: whenever a jump at time \( t \) occurs from \( u \) in one of the systems, with \( X^x_{r-} - r \leq u < X^x_{r+} + r \), then \( X^x_t = X^x_{r-} + 2r \). Similarly \( Y^y_t \) starts at \( y \), and \( Y^y_t = Y^y_{r-} - 2r \) if a jump occurs at time \( t \) from \( Y^y_{r-} - r < u \leq Y^y_{r+} + r \). The rate at which \( X \) jumps is bounded by a constant depending on \( r \) and \( \sup_{z, \eta} c(0, z, \eta) \). Therefore the number of jumps of \( X \) is bounded by some Poisson process, and similarly for \( Y \). Finally, by construction, \( \eta \) and \( \zeta \) coincide at times \( t \) and sites \( X^x_t \leq u \leq Y^y_t \) so long as

\[
Y^y_t - X^x_t \geq 2r. \quad \square
\]

**Proof of Lemma 3.2.** For \( x \in \mathbb{Z} \), we let \( S^x_t \) denote the number of particles to the left of \( x \) including \( x \). Since jump rates are bounded and jumps have bounded range, the total rate at which particles leave (resp. enter) \( (-\infty, x] \cap \mathbb{Z} \) is bounded. Therefore, there exists a constant \( M > 0 \) and two Poisson processes \( L^x \) and \( E^x \) such that

\[
-M(L^x_t - L^x_s) \leq S^x_t - S^x_s \leq M(E^x_t - E^x_s),
\]

for all \( s, t > 0 \) with \( s \leq t \), where \( M \) is independent of \( x \) by translation invariance. Let \( X_t \) and \( Y_t \) denote the positions of the leftmost and rightmost particle. Proceeding as in the proof of Lemma 3.1, \( X_t \) and \( Y_t \) can be bounded, respectively, by a backward and
forward moving Poisson process. Let

$$\Omega^N_x = \left\{ \sup_{t \in [0, T]} |S^N_{Nt+N\varepsilon} - S^N_{Nt}| > N\delta \right\}.$$  

Denote by $\Omega^N$ the event in Lemma 3.2, and set

$$\Omega'_N = \{ \forall t \in [0, T], \max(|X_{Nt}|, |Y_{Nt}|) \leq a + CNt \}.$$  

Using large deviation estimates for the Poisson process, we can choose $C$ large enough and $\varepsilon$ small enough such that $P(\Omega'N)$ converges to 1 and $P(\Omega^N_x) \leq Ce^{-Nt/C}$. The result then follows from

$$\Omega^N \subset (\Omega'^N)^c \cup \bigcup_{x:|x| \leq a+CNt} \Omega^N_x.$$  

References


